The Generalized Tonnetz

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Abstract This article relates two categories of music-theoretical graphs, in which points represent notes and chords, respectively. It unifies previous work by Brower, Callender, Cohn, Douthett, Gollin, O'Connell, Quinn, Steinbach, and myself, while also introducing new models of voice-leading structure—including a three-note octahedral Tonnetz and tetrahedral models of four-note diatonic and chromatic chords.

Music theorists typically represent voice leading using two different kinds of diagram. In *note-based graphs*, points represent notes, and chords correspond to extended shapes of some kind; the prototypical example is the Tonnetz, where major and minor triads are triangles, and where parsimonious voice leadings are reflections ("flips") preserving a triangle's edge. In *chord-based graphs*, by contrast, each point represents an *entire sonority*, and efficient voice leading corresponds to short-distance motion in the space, typically along an edge of a lattice. This difference is illustrated in Figure 1, which offers two perspectives on the same set of musical possibilities: on the top, we have the traditional note-based Tonnetz, while on the bottom we have Jack Douthett and Peter Steinbach's (1998) chord-based "chicken-wire torus."[1] These figures both represent single-step (or "parsimonious") voice leading among major and minor triads and are "dual" to each other in a sense that will be discussed shortly.

In *A Geometry of Music* (Tymoczko 2011), I provide a general recipe for constructing chord-based graphs, beginning with the continuous geometrical spaces representing *all* n-note chords and showing how different scales determine different kinds of cubic lattices within them. I also showed that nearly even chords (such as those prevalent in Western tonal music) are represented by three main families of lattices. Two of these are particularly useful in analysis: the first consists of a circle of n-dimensional cubes linked by

Thanks to Richard Cohn and Gilles Baroin for helpful comments.

1 The chicken-wire torus was introduced in Douthett and Steinbach 1998. There are many different orthographical variants of the traditional Tonnetz, depending on how one orients the axes; for a survey, see Cohn 2011a. For present purposes, these are all equivalent.

Journal of Music Theory 56:1, Spring 2012
DOI 10.1215/00222909-1546958

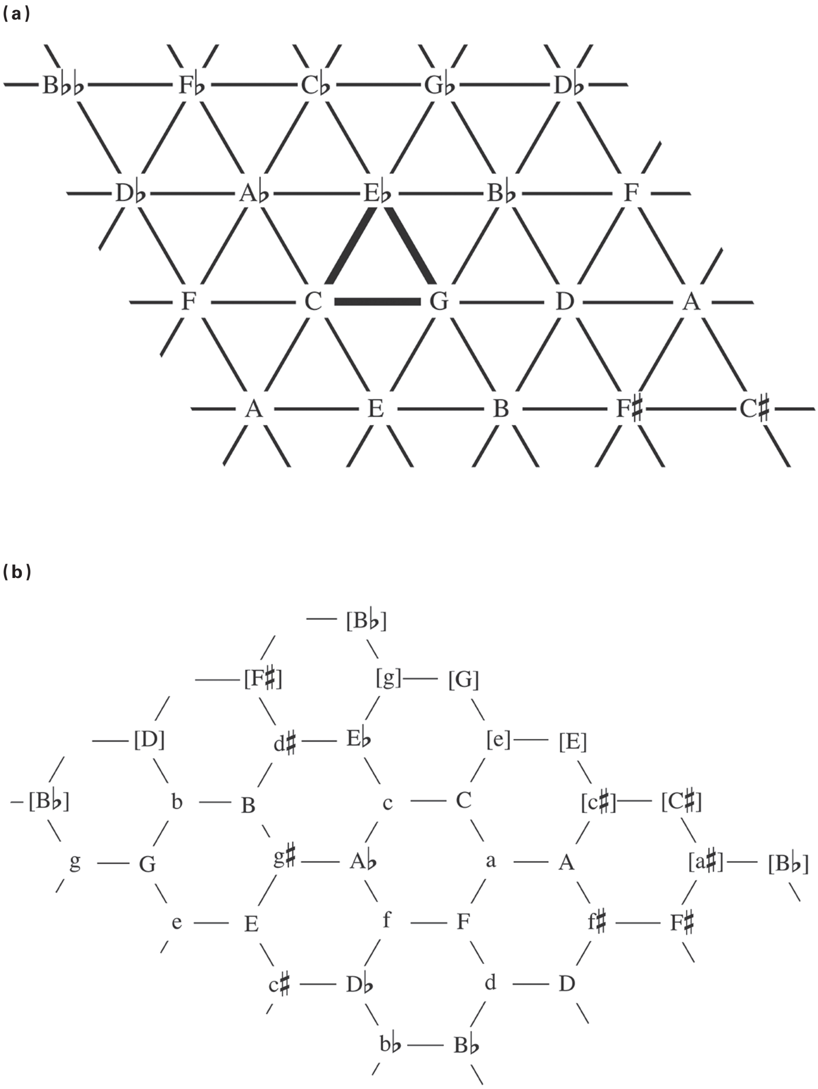


Figure 1. Two versions of the Tonnetz. (a) The note-based version, in which points represent notes and triangles represent chords. (b) Its geometrical dual, called the "chicken-wire torus" by Douthett and Steinbach (1998). Here, points represent chords and edges represent single-step voice leading.

shared vertices; the second consists of a circle of n-dimensional cubes linked by shared *facets* (the third does not often appear in practical contexts, and can be safely ignored).[2] What results is a systematic perspective on the full family of chord-based graphs.

The question immediately arises whether we can provide a similarly systematic description of the note-based graphs. What note-based construction represents efficient voice leading among nearly even four-note chords in the chromatic scale? What about nearly even four-note chords in the diatonic scale? How can we generalize the familiar Tonnetz to arbitrary chords within arbitrary scales? Is there a note-based graph for every chord-based graph? Is one or the other type of graph more useful for particular applications?

The purpose of this article is to answer these questions by providing a recipe for constructing generalized note-based graphs, or Tonnetze. Along the way we will encounter some surprising facts:

- The Tonnetz, while apparently a two-dimensional structure, can also be understood as a *three-dimensional* circle of octahedra linked by shared faces. The shared faces represent augmented triads, which do not appear on the traditional Tonnetz. The two versions of the Tonnetz are graph-theoretically identical but geometrically (and topologically) distinct.
- The seventh-chord analogue to the traditional Tonnetz can be depicted as a series of nested tetrahedra, each containing the notes of a diminished-seventh chord. This figure represents efficient voice leading among diminished, half-diminished, dominant seventh, minor-seventh, and French sixth chords.
- The traditional Tonnetz is often described as a torus, or a “circle times a circle.” However, the more general description is the “twisted product of an $(n-2)$-dimensional sphere with a circle.” It just so happens that in the three-note case, the one-dimensional sphere is itself a circle, potentially misleading theorists into thinking that higher-dimensional Tonnetze are also toroidal.
- Any sufficiently large note-based graph will inevitably contain either “flip restrictions” or “redundancies”—that is, the graph will either contain “flips” that represent *nonstepwise* voice leadings or multiple representations of the same chord. The traditional Tonnetz is unusual in that it lacks both features.
- Chord-based voice-leading graphs are associated with note-based Tonnetze by the geometrical property of *duality*. However, the duality

2 This type of graph occurs only when the number of notes in the chord is less than half the size of the scale, and shares a common factor with it, but does not divide the size of the scale exactly. The only time we would encounter these graphs, for scales smaller than fourteen notes, is when exploring four-note chords in a ten-note scale, hardly an everyday occurrence.

relation obtains not between graphs considered as unified wholes, but rather between their cubic and octahedral components.

From a theoretical point of view, the last point is the crucial one. The most natural route to the generalized Tonnetz begins with the chord-based lattices described in *A Geometry of Music*. These are typically arrangements of n-dimensional cubes. We can replace each n-cube with its *geometrical dual*, producing a collection of "generalized octahedra." These generalized octahedra then need to be rotated or reflected before they can be glued together to form the note-based analogue to the original chord-based graph.

Geometrical investigations of chordal voice leading began with the note-based Tonnetz, a structure that was originally devised to represent purely acoustical relationships among notes.[3] But as the geometrical approach matured, it gradually moved toward chord-based graphs, which are more easily generalized to a broader range of musical circumstances. Having understood these chord-based structures, we can now complete the circle, returning to the note-based graphs that started the investigation. Thus, more than two decades after the beginnings of neo-Riemannian theory, we are poised to understand the Tonnetz in a deeper and more principled way.

1. Mathematical Background

This section reviews some basic mathematical material, beginning with elementary geometrical terminology and proceeding to describe the duality of the hypercube and the cross-polytope. I will try to be informal and intuitive, in keeping with my goal of remaining comprehensible to readers who are musicians first and foremost. This is consistent with my philosophy that music theory is an applied discipline in which mathematics is a tool rather than an end in itself.[4]

One word of warning: when doing higher-dimensional geometry, it is often necessary to prioritize algebra over direct visualization. In large part, geometry is a matter of grasping patterns that repeat themselves in increasing dimensions, with algebraic relations providing our best guide as to which properties do, in fact, generalize. Thus, rather than struggling to construct a vivid picture of the seven-dimensional cross-polytope, one should instead concentrate on the generic properties shared by *all* cross-polytopes, contenting oneself with visualizing only the lower-dimensional examples.[5] That said, music can sometimes be a useful tool for picturing higher-dimensional rela-

3 It was Richard Cohn (1997) who pioneered the use of geometrical graphs, and in particular the note-based Tonnetz, to represent chordal voice leading. Some earlier work, such as Roeder 1984 and 1987, used geometry to represent voice leading among *set-classes*. For a brief history of the development of geometrical models of voice leading, see Tymoczko forthcoming.

4 Readers who pine for mathematical rigor will likely be able to generate proofs from the following informal exposition.

5 A substantial number of blind mathematicians are geometers (Jackson 2002). One hypothesis is that blindness can be helpful, insofar as it reduces the reliance on quasi-visual pictures.

tionships; for example, readers who have finished this article will have no trouble imagining the seven-dimensional cross-polytope as a certain collection of relations between two completely even seven-note scales.

Basic terminology

In plane Euclidean geometry, a *polygon* is a two-dimensional plane figure bounded by a closed sequence of line segments. A *vertex* is a point belonging to two adjacent line segments. A polygon is said to be *convex* if its interior contains every line segment between any two points of the polygon. (Convex polygons have internal angles less than or equal to 180°.) These definitions can be generalized to higher dimensions: the three-dimensional analogue of a polygon is a *polyhedron*, while the n-dimensional analogue is a *polytope*. A polyhedron is bounded by polygonal *faces* (dimension 2) that intersect at linear *edges* (dimension 1), which in turn intersect at points called *vertices* (dimension 0). An n-dimensional polytope is bounded by *facets* that are all $(n - 1)$-dimensional polytopes, themselves intersecting to form $(n - 2)$-dimensional polytopes (*ridges*) that intersect to form $(n - 3)$-dimensional polytopes (*peaks*) . . . all the way down to two-dimensional faces, one-dimensional edges, and zero-dimensional vertices. The term *codimension* is sometimes useful: if W is a subspace of V, then the codimension of W in V is the dimension of V minus the dimension of W (that is, the number of "extra" dimensions in V not taken up by W). A facet always has codimension 1, a ridge has codimension 2, and so on.

A *hyperplane* is an infinite flat space of codimension 1. In $(n + 1)$-dimensional Euclidean space, the n-dimensional *sphere* (n-sphere) is the set of points equidistant from the origin; the n-dimensional *ball* consists of all points less than or equal to a certain distance from the origin. (It is a "filled-in" sphere, the union of a sphere and its interior.) *Topological equivalence* can be understood as "equivalence to within stretching": two shapes are topologically equivalent if one can be smoothly deformed into the other without tearing or gluing. (Imagine the shapes being made out of infinitely flexible rubber.) All convex n-dimensional polytopes are topologically equivalent to the n-dimensional ball.

It is important to distinguish a space's *intrinsic* and *extrinsic* dimensionality. Intuitively, the former refers to the number of perpendicular directions in which one can move, at any point in the space; the latter refers to the way the space is embedded in some other, higher-dimensional space. For example, a circle is intrinsically one-dimensional, since at any point one can move only clockwise or counterclockwise.[6] *Intrinsically*, then, the circle has one dimension; we typically conceive of it *extrinsically* as being embedded in a two-dimensional space (in much the same way the surface of a globe is

6 Counterclockwise is the opposite of clockwise and does not count as a separate (perpendicular) direction.

intrinsically two-dimensional, though often represented in three dimensions).[7] Mathematicians are typically concerned with the intrinsic rather than extrinsic dimension of a space. In this article, however, the issue of dimensionality will be rather subtle, since we will be considering graphs that are embedded within larger geometrical spaces. Thus, we will find ourselves wondering whether it is better to model the augmented triad on the familiar Tonnetz as a circle of intrinsic dimension 1, or as a triangle embedded in a three-dimensional space, having extrinsic dimension 2.

Related to this is the contrast between graph theory and geometry. From a formal point of view, a *graph* is a very abstract structure—a collection of points (or *vertices*) along with a series of connections between them (*edges*), not necessarily embedded in any larger space. Graphs do not have straight lines, angles, or any determinate topology.[8] In what follows, however, I will sometimes use the term *graph* to refer to a series of vertices connected by line segments *and contained within some continuous geometrical space*. This is because chord-based graphs are typically embedded within the continuous spaces representing all possible *n*-note chords. As we will see, it is sometimes useful to take this point of view with respect to the note-based graphs as well.

The dual polytope

A convex polytope (polyhedron, polygon) can be associated with another polytope known as its *geometrical dual*. For every *facet* of the original polytope (= region of codimension 1), the dual has a *vertex*. (In many contexts, we can imagine this vertex to be situated in the center of the original facet.) Two vertices in the dual are linked by an *edge* if they are associated with facets that intersect in a *ridge* (= region of codimension 2). (Thus, for every ridge in the original space, the dual has an edge.) Figure 2 shows that the dual of a square is another square, while the dual of a cube is an octahedron. The dual of the octahedron is a cube, illustrating the general principle that every polytope is its dual's dual. The square is "self-dual" since its dual is another square. The triangle and tetrahedron are also self-dual.

Hypercubes, cross-polytopes, duality, and simplexes

The duality relation between cubes and octahedra can be extended to arbitrary dimensions, with an *n*-dimensional cube being known as the "hypercube," the *n*-dimensional octahedron being the "cross-polytope," and the two structures being dual to each other. Mindful of my earlier warnings against

7 Note that if we were to "zoom in" to a very small region of a circle (or sphere), the curvature would gradually disappear, and the space would seem more and more like a line (or plane).

8 Lewinian "node/arrow systems," though more structured than graphs, are similarly abstract (Lewin 1987).

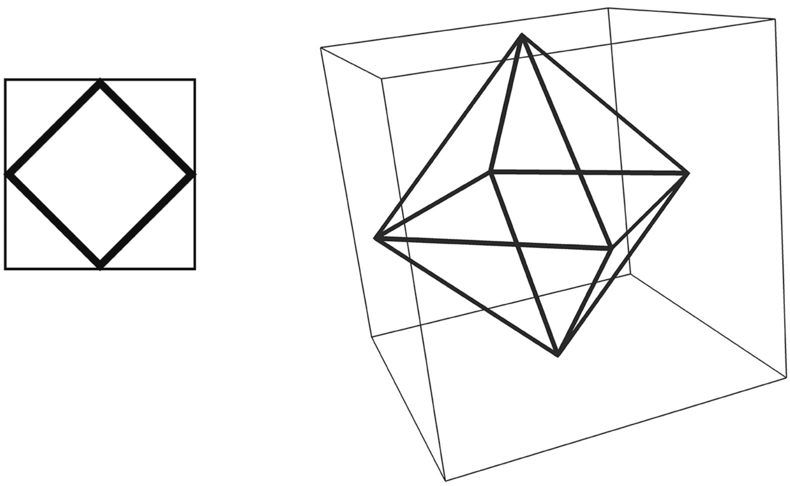


Figure 2. The dual of a convex polytope is another polytope that has a vertex for every face of the original. The dual of a square is another square, while the dual of a cube is an octahedron.

visualization, we will investigate the relationship by introducing algebraic coordinates that can be interpreted musically.

In n-dimensional Cartesian space, we can form an n-dimensional hypercube by considering the figure whose vertices are $(\pm1, \pm1, \pm1, \ldots, \pm1)$.[9] Its dual n-dimensional cross-polytope has vertices whose coordinates are the permutations (reorderings) of $(\pm1, 0, 0, \ldots, 0)$. These coordinates are convenient insofar as each facet of the hypercube will be bounded by vertices that share a single number in some particular order position (Figures 3 and 4). As is clear from the illustrations, any cubic facet's shared coordinate is the nonzero coordinate of the associated vertex in the dual cross-polytope.

It is easy to see that the facets of an n-dimensional hypercube are $(n-1)$-dimensional hypercubes.[10] The facets of an n-dimensional cross-polytope are neither cubes nor cross-polytopes, but are instead $(n-1)$-dimensional *simplexes.* A simplex is a generalized triangle or tetrahedron: an n-dimensional simplex is bounded by $n + 1$ vertices, not all in the same hyperplane, with edges connecting all vertices. It is called a "simplex" because it is the n-dimensional polytope with the fewest vertices; in that sense, it can be said to be as simple as possible.[11] The cross-polytope whose vertices are permutations of $(\pm1, 0,$

9 That is, all possible combinations of the values +1 and –1.

10 If we fix any one coordinate, then we are left with an $(n-1)$-dimensional figure whose vertices are $(\pm1, \pm1, \pm1, \ldots, \pm1)$. This is the equation for an $(n-1)$-dimensional hypercube.

11 In n-dimensional space, there is an important $(n-1)$-dimensional simplex whose vertices are the permutations of $(1, 0, 0, \ldots, 0)$. (This is in fact one of the facets of the cross-polytope described in the main text.) All points on this simplex have coordinates that sum to the value 1. The simplex can therefore be taken to represent the different

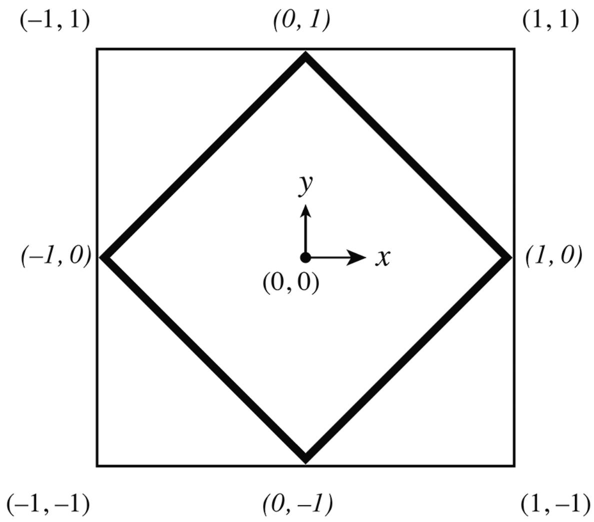


Figure 3. If the square has vertices whose coordinates are (±1, ±1), then the dual has vertices whose coordinates are (±1, 0) and (0, ±1).

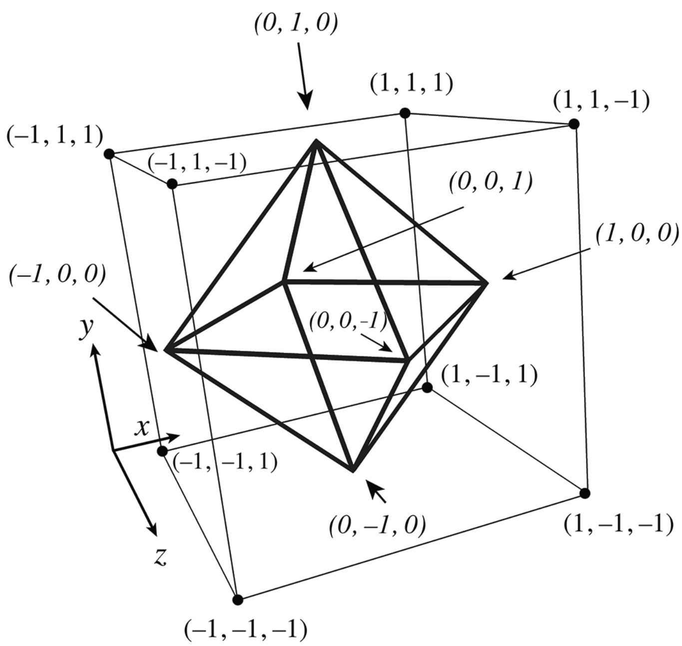


Figure 4. If the cube has vertices whose coordinates are (±1, ±1, ±1), then the dual has vertices whose coordinates are (±1, 0, 0), (0, ±1, 0), and (0, 0, ±1). Each of the octahedron's vertices is therefore situated in the middle of one of the square's faces.

0, . . . , 0) has facets that are bounded by a collection of *n* vertices no two of which are nonzero in the same coordinate: thus each facet is determined by choosing signs for the collection (±1, 0, 0, . . . , 0), (0, ±1, 0, . . . , 0), . . . , (0, 0, 0, . . . , ±1).[12] This cross-polytope can be thought of as having a simplicial "top facet" whose coordinates are the reorderings of (+1, 0, 0, . . . , 0) and a simplicial "bottom facet" whose coordinates are the reorderings of (–1, 0, 0, . . . , 0). (Note that on Figure 3 the "top facet" is on the upper right, while on Figure 4 it is on the upper right rear. It would be possible to rotate the coordinate axes so that this face did indeed appear to be the "top" of the figure.) The remaining facets combine vertices from the top and bottom facets in an appropriate way. Edges connect every vertex on the top facet to all the vertices on the bottom facet *except* the one that is nonzero in the same coordinate.

Musically, we will interpret the hypercube (±1, ±1, ±1, . . . , ±1) as representing a chord-based graph that records all possible sequences of lowerings that take an ordered *n*-note chord to the chord a step below it. Let any +1 coordinate refer to one of the chord's original notes, and –1 refer to the note a step below. Figure 3 shows that we can move from (+1, +1) to (–1, –1) by passing through either (+1, –1) or (–1, +1); similarly, Figure 4 shows that we can move from (+1, +1, +1) to (–1, –1, –1) by passing through intermediate vertices such as (+1, +1, –1) and (+1, –1, –1). The "dual" cross-polytope, whose vertices are the permutations of (±1, 0, 0, . . . , 0), can be interpreted as a *note-based graph* recording the same information. We begin with the original chord at the top facet (that is, the facet whose vertex coordinates are all +1, corresponding to the original, "unlowered" form of each note). Each single-step lowering is represented by a "simplex flip" that replaces a top-facet vertex with its bottom-facet analogue (that is, the vertex with –1 in the same order position). For example, on Figure 3 we can flip the upper-right edge onto the upper left around the top vertex, lowering the first coordinate in the process; on Figure 4, we can flip the upper-right-rear triangle onto the upper-right front, lowering the third coordinate. We can continue to flip, replacing +1 vertices with their –1 counterparts, until we have reached the bottom facet, where no more lowerings are available.

This interpretations will be central to the rest of this article, so readers are encouraged to study Figures 3 and 4 carefully. Alternatively, you may

ways of dividing a fixed quantity of stuff into *n* different piles; for instance, the line segment from (1, 0) to (0, 1) is a one-dimensional simplex that can represent the results of a two-party election, while the triangle whose vertices are (1, 0, 0), (0, 1, 0), and (0, 0, 1) is a two-dimensional simplex that can represent the results of a three-party election—or the relative size of the three intervals in a three-note transpositional set-class, or a three-pulse rhythm in a measure of fixed size.

12 For instance, the facets of the octahedron in Figure 3 are bounded by sets such as (1, 0, 0), (0, 1, 0), and (0, 0, 1). There are no facets that contain pairs such as (1, 0, 0) and (–1, 0, 0), as these correspond to opposite faces of the associated cube.

prefer to return to these ideas after the following section, which clarifies the musical relevance of the points just discussed.

2. Music-Theoretical Background

Now we need to review some basic theoretical material from chapter 3 of *A Geometry of Music*. We will start with a general discussion of the standard Tonnetz, then turn to the role of hypercubes in the continuous chord spaces, and end with the two main families of chord-based voice-leading lattices.

The multivalent Tonnetz

Since the Tonnetz is our central example of a note-based graph, it pays to consider it carefully. In particular, it is important to realize that the Tonnetz has a triply ambiguous status as a representation of *acoustic*, *voice-leading*, and *common-tone* relationships. From a historical point of view, the acoustic aspect is primary: Leonhard Euler originally designed the structure so that maximally consonant intervals—the perfect fifth and major third—corresponded to the graph's edges. The Tonnetz's second role, as a representation of *single-step voice-leading relationships among major and minor triads*, became important in Richard Cohn's early papers (see esp. Cohn 1996, 1997). It is this second aspect that will concern us here: we will be trying to generalize the Tonnetz *considered as a graph of efficient voice-leading possibilities* rather than as a representation of acoustical relationships.

There is, however, one complication. Cohn's early work, like neo-Riemannian theory more generally, often conflated voice-leading efficiency and common-tone retention. This can be seen in Cohn's practice of measuring voice-leading distance in terms of "Tonnetz flips" or "parsimonious voice-leading moves."[13] While it might seem that this strategy would produce an effective measure of voice-leading size, this is not the case: the voice leading (C, E, G) → (C, F, A) is smaller by this measure than the voice leading (C, E, G) → (C, F, A♭), whereas from the voice-leading perspective the opposite is true. Moreover, common-tone retention does *not* necessarily produce the most efficient voice leading between chords: the common tone–preserving (E4, F4, G4) → (E4, F4, D4) has a voice crossing and is, for almost every standard measure of voice-leading size, less efficient than (E4, F4, G4) → (D4, E4, F4), which has no crossings and preserves no common tones.[14]

The crucial point is that the three conceptions of the Tonnetz generalize in different ways. The most natural generalization of the acoustic Tonnetz is a structure in which additional axes represent additional consonant intervals such as the octave (in the work of Longuet Higgins) or the just minor

13 See Cohn 1997, which categorizes progressions by the number of parsimonious voice-leading moves ("binary," "ternary," etc.).

14 Tymoczko 2006, 2008b, and 2011, appendix C. See also Hall and Tymoczko 2012.

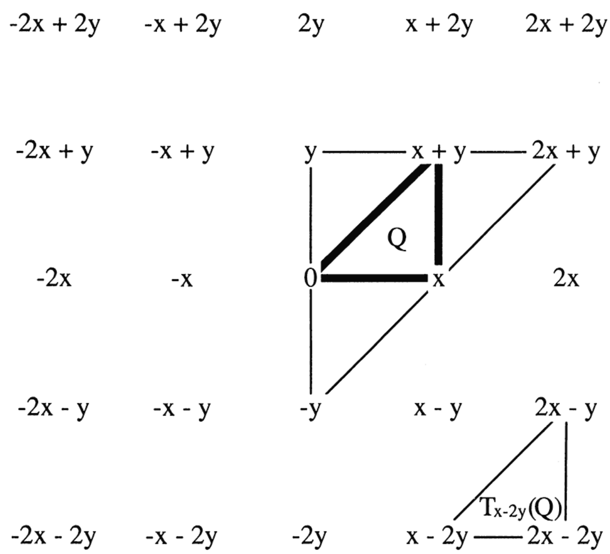


Figure 5. Cohn's generalized Tonnetz describes a chord (0, *x*, *x* + *y*). Each trichord shares an edge (and hence two common tones) with three of its inversions.

seventh (in the work of some contemporary tuning theorists) (Tymoczko 2009a). The most natural generalization of the common-tone Tonnetz is the one that Cohn constructed in his influential 1997 article "Neo-Riemannian Operations, Parsimonious Trichords, and Their 'Tonnetz' Representations," shown here as Figure 5. (Note that this generalization cannot be extended to chords of arbitrary size.) By contrast, the "voice-leading Tonnetz" is one of a large family of note-based graphs that will be explored below.

Cubic geometry in chord spaces

Chapter 3 of *A Geometry of Music* provides a detailed, user-friendly introduction to the continuous geometrical spaces representing all possible chords. Here I very briefly review the essentials, trusting that readers will consult the book in the event of any confusion. Chords live in quotient spaces or orbifolds, arising when we "glue together" ordered pitch sequences that represent the same set of pitch classes. We start with n-dimensional Cartesian space, $\mathbb{R}^n$, which can be pictured as an infinite space with n linear axes, all at right angles to one another. Points in this space are represented by n-tuples of real numbers, one for each axis or voice, with each coordinate representing the pitch sounded by that particular voice. To form a space of musical chords, we need to ignore octave and order. We ignore octave by considering the coordinates modulo 12, or in other words, gluing together the points $(\ldots, x, \ldots)$ and $(\ldots, x + 12, \ldots)$. This transforms our space into the n-dimensional torus

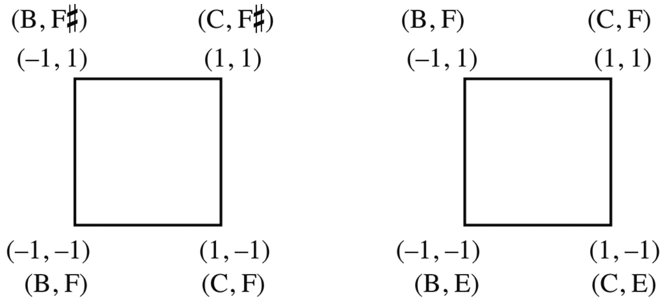


Figure 6. A square represents all the ways in which one can lower the notes of a two-note chord, eventually producing the chord a scale-step below. This is true whether we are working in the chromatic scale (left), diatonic scale (right), or any other scale.

$\mathbb{T}^n$, or *ordered pitch-class space*. (A torus can be thought of as an *n*-dimensional space with *circular* axes, all at right angles to one another.) To ignore order, we glue together all points whose coordinates are related by permutation. This produces a twisted *n*-dimensional donut, known to mathematicians as $\mathbb{T}^n/\mathcal{S}_n$.[15] This space contains *singular points* that act like mirrors, with line segments appearing to "bounce off" these singularities like billiard balls reflecting off the edges of a pool table.

The global structure of these spaces is not important in what follows. For us, the crucial fact is that the spaces admit "locally valid" coordinate systems in which the orthogonal axes correspond to motion in the individual musical voices.[16] (Mathematical readers will note that this follows from their construction: we began with $\mathbb{R}^n$, in which each coordinate represented a different voice; our various "gluings" changed the space only at a few singular points, with the bulk of the space remaining locally isomorphic to regions of the original.) Discrete chord-based voice-leading lattices are typically constructed from *hypercubes* in which the various spatial dimensions represent single-step motion in each of a chord's various voices.[17] A single hypercube will represent the different ways of lowering (or raising) the notes of one particular chord by step. This is illustrated in Figures 6 and 7, which show the two- and three-note diatonic and chromatic cases. Note that the basic structure of the diatonic and chromatic graphs is identical, with the only difference being the location of the transpositional relationships; for instance, in

15 This space has one facet; the Möbius strip has one edge; three-note chord space, a twisted triangular donut, has one face, and so on. Points represent chords, while line segments represent *voice leadings* or ways of moving from one chord to another.

16 The coordinate system is only "locally" valid since the space has a "nontrivial holonomy," which means that a circular path can rotate or reflect the axes.

17 Actually, this is true only when we are considering *n*-dimensional lattices in *n*-dimensional space. There are some cases when the lattice will have lower dimension than the ambient space, and the faces will fix multiple pitch classes. We will return to this point below.

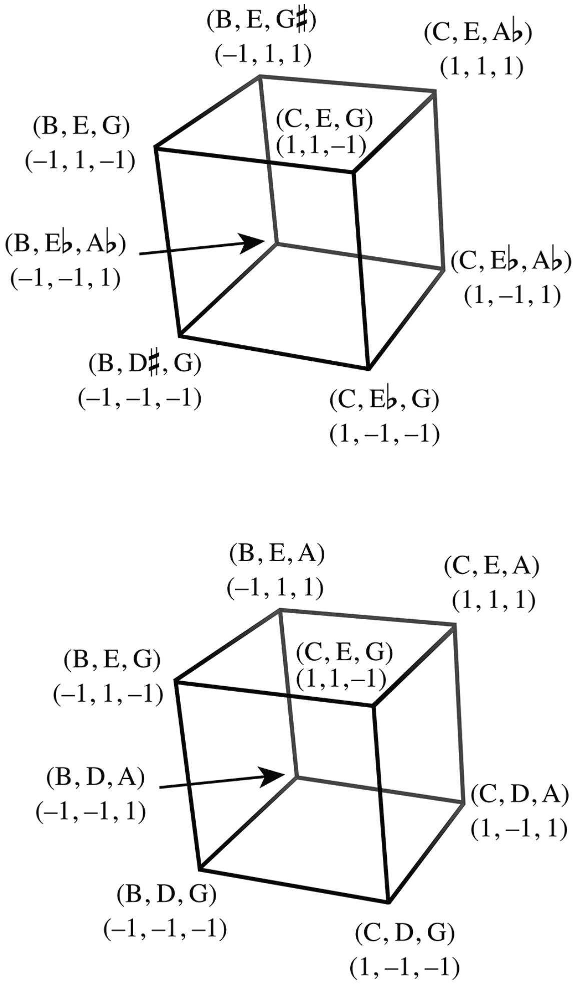


Figure 7. A cube represents all the ways in which one can lower the notes of a three-note chord, eventually producing the chord a scale step below. This is again true no matter what scale we are working in.

Figure 6, the chromatic graph has two tritones and two perfect fifths, whereas the diatonic graph has three fourths (all related by diatonic transposition) and one third. Likewise, in Figure 7, the chromatic cube has two augmented triads, three major triads, and three minor triads; the diatonic case has four triads, two fourth chords, and two incomplete sevenths. Nevertheless, *all* the graphs show the various ways of stepwise lowering the notes of the chord on the upper right (or upper-right rear) of the figure.

Figure 8 shows that there is another way of conceiving the coordinate systems in our space: instead of understanding the various directions as *mov-*

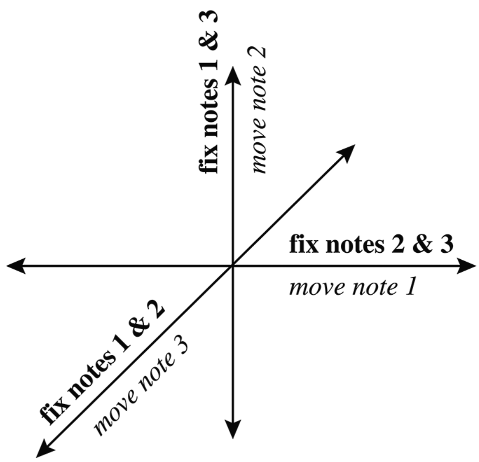


Figure 8. In *n*-dimensional chord space, we can find a coordinate system such that each direction corresponds to motion in a particular voice. Alternatively, we can think of each direction as holding all but one voice constant. It follows that $n - 1$ of these axes define a hyperplane holding a single note constant.

ing one voice, we can conceive of them as *holding fixed all but one voice*. It follows that, at any point in the space, the hyperplane defined by all but one of our axes will be associated with a single fixed pitch class. This in turn means that the *facets* of our voice-leading cubes will each fix some particular pitch class, common to all the chords on its vertices. (This is also true of the hyperplane containing that facet.) When we transform such a cube into its geometrical dual, each facet in the original cube becomes a point in the dual cross-polytope. It follows that the vertices of the dual polytope represent particular *pitch classes*. Chords in the dual are now represented by the cross-polytope's simplicial facets. (Since every polytope is the dual of its dual, these facets in turn correspond to vertices in our original, cubic, chord-based graph, so everything works out as expected.) In the dual graph, single-step voice leading is represented by a "simplex flip" that reflects one simplex into another through a common ridge.[18]

Thus, if we start with an *n*-dimensional cube in *n*-note chord space, representing single-step voice-leading in each voice, we can use duality to form a Tonnetz analogue, a "generalized octahedron" (cross-polytope) in which vertices represent pitch classes and facets ("generalized triangles" or simplexes) represent chords. Efficient voice leading is now represented by

18 That is, reflections preserving a common ridge of the original cross-polytope, which is to say a facet of the simplical facet.

"simplex flips" that transform one facet of the octahedron into another that shares a common ridge. The only difficulties are (1) determining how these various "generalized octahedra" are to be glued together and (2) describing the resulting structures.

The two families of lattices

In §3.11 of *A Geometry of Music*, I showed that just three families of lattices represent the most nearly even *n*-note chords in any scale. Only two of these play a major role in analysis. In the first family, the size of the chord evenly divides the size of the scale, and we have a circle of *n*-dimensional cubes linked by shared vertices (Figure 9).[19] In the second family, the size of the chord is *relatively prime* to the size of the scale, and we have a circle of cubes linked by shared facets. The most interesting difference between the two families is that in the first, the dimension of the lattice is determined by the size of the chords we wish to represent (to represent two-, three-, and four-note chords, we need two-, three-, and four-dimensional lattices, respectively); whereas in the second family, the dimension is controlled by *the number of chord types we wish to represent*. Thus, we can use a one-dimensional graph to represent the voice-leading relationships among maximally even chords (no matter how many notes they have!), a two-dimensional graph to represent voice leading among the two most even types of chord, a three-dimensional graph to represent the three most even types of chord, and so on.

The basic principles here are relatively simple. If the size of a chord is relatively prime to the size of a scale, the maximally even chord is a "near interval cycle" all but one of whose intervals are the same, with the unusual interval being just a scale step different from the others (see Clough and Myerson 1985; Clough and Douthett 1991). It follows that we can use this chord to create a "generalized circle of fifths"—a circle of transpositionally related chords, each connected by single-step voice leading to its neighbors (Figure 10).[20] To include the second most even type of chord, we *reverse the order of every pair of adjacent voice leadings in the "generalized circle of fifths."* Figure 11 represents this geometrically, arranging the generalized circle of fifths in a zigzag. (In this arrangement, reordering a northeast-then-southeast move involves moving southeast-then-northeast, and so on.) To represent the *third* most even type of chords, we begin with a zigzag through *three* dimensions, reordering every triple of adjacent voice leadings in the circle of fifths (Figure 12). Remark-

19 A hypercube represents all the ways of systematically lowering the notes of a chord by step; if the size of the chord divides the size of the scale, then the scale contains a *perfectly even* chord that divides the octave into *n* equal parts. This chord is located at the cubes' shared vertices; the remaining vertices are generated by all the different ways of successively lowering its notes by step. Clearly, no two perfectly even chords will have common tones.

20 Clough and Myerson (1985) use the term "generalized circle of fifths" to refer to what I have just called a "near interval cycle," represented as a circular *note-based graph*. By contrast, I am using the term to refer to a circular *chord-based graph* in which transpositionally related chords are linked by single-step voice leading.

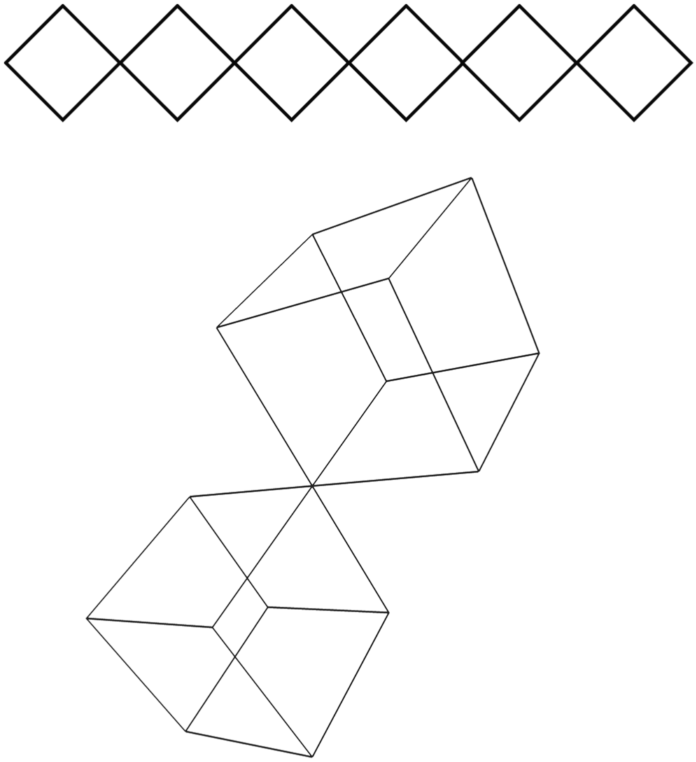


Figure 9. When the size of the chord divides the size of the scale, the most nearly even chords are represented by a circle of *n*-dimensional cubes linked to their neighbors by a shared vertex. Here are the one- and two-dimensional cases.

ably, this procedure suffices to generate our second family of lattice. Thus, the abstract graph in Figure 11 can be filled in by fifths, triads, or seventh chords in the diatonic scale, and pentatonic or diatonic collections in the chromatic scale, while that in Figure 12 can be filled in by diatonic triads, octatonic triads, or familiar seven-note scales.

Readers who want further information are directed to §3.11 of *A Geometry of Music*. Here the important point is that both families of lattice are constructed from (hyper)cubes joined together in some way, either by shared vertices (Figure 9) or by shared facets (Figures 11 and 12). We have already seen that we can convert chord-based graphs into note-based graphs by replacing hypercubes with their dual cross-polytopes. Thus, we simply need to determine how the cross-polytopes fit together.

3. Constructing the Note-Based Lattices

We will now convert familiar chord-based graphs into note-based structures analogous to the Tonnetz. Again, the basic strategy will be to replace each of

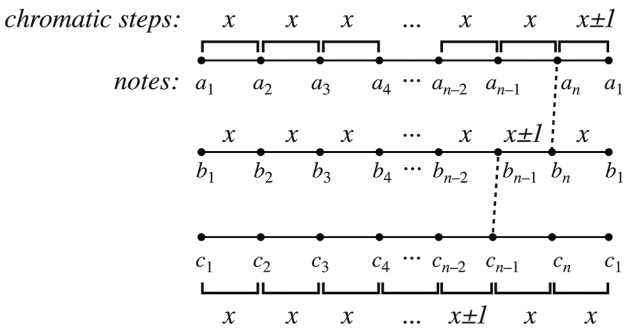


Figure 10. **When the size of the chord is relatively prime to the size of the scale, the most nearly even chord is a "near interval cycle"—a circle of *n* intervals, all but one the same size, with the outlier being just a scale step larger or smaller. Given any such chord, we can construct a "generalized circle of fifths," or circle of single-step voice leadings connecting transpositionally related chords. The circle is formed by moving the position of the unusual interval. Note that this is a note-based graph, in contrast to the surrounding figures.**

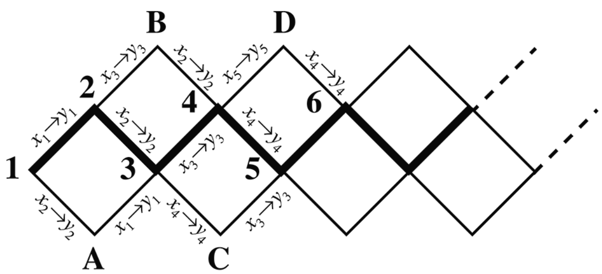


Figure 11. **Given a "generalized circle of fifths" (1 → 2 → 3 → · · ·), we can graph single-semitone voice leadings among the two most even chord types by scrambling every pair of adjacent voice leadings along the circle of fifths. This schematic chord-based graph can therefore represent voice-leading relations among diatonic triads and fourth chords, diatonic and acoustic scales, diatonic fourths and thirds, and so forth.**

the (hyper)cubes in a chord-based graph with its geometrical dual, connecting adjacent cross-polytopes as required. For clarity, we will treat each family of graph separately, beginning with the two- and three-dimensional cases before turning to higher dimensions.

Note that while the discussion focuses on chords contained within chromatic and diatonic scales, the underlying ideas are inherently more general than that. What is important is not the particular structure of any particular scale, but simply the relation between the size of the chord and the size of the

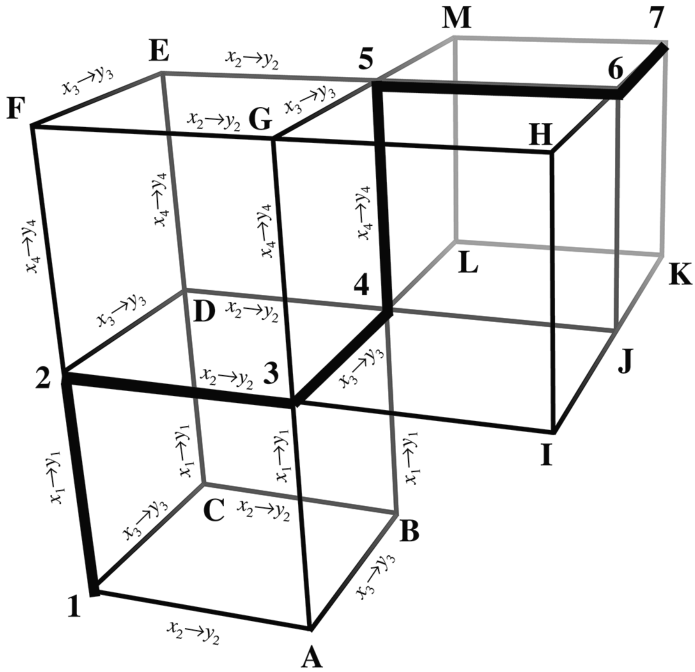


Figure 12. Given a "generalized circle of fifths" (1 → 2 → 3 → · · ·), we can graph single-semitone voice leadings among the three most even chord types by scrambling every three adjacent voice leadings along the circle of fifths.

scale—in particular, whether one number divides the other, or whether the two numbers are relatively prime. Thus, the graph of nearly even four-note octatonic chords has the same basic structure as the graph of nearly even four-note chromatic chords, since 4 divides both 8 and 12. In much the same way, the graph of nearly even five-note diatonic chords is structurally similar to the graph of nearly even five-note chromatic chords, since 5 is relatively prime to both 7 and 12. Readers interested in more exotic cases should therefore be able to generate the relevant graphs from the following examples.

First family, two dimensions

Figure 13a shows single-semitone voice leading between perfect fourths (= fifths) and tritones in the chromatic scale; it is a circle of squares, each linked to its neighbors by a shared vertex. Tritones are on the shared vertex with fourths on the top and bottom vertices. The two 45° axes (northwest/southeast and northeast/southwest) represent motion in the individual voices. For example, at {B♭, E}, motion along one diagonal moves B♭ up and down by a semitone, keeping E fixed, while the other diagonal moves E up and down,

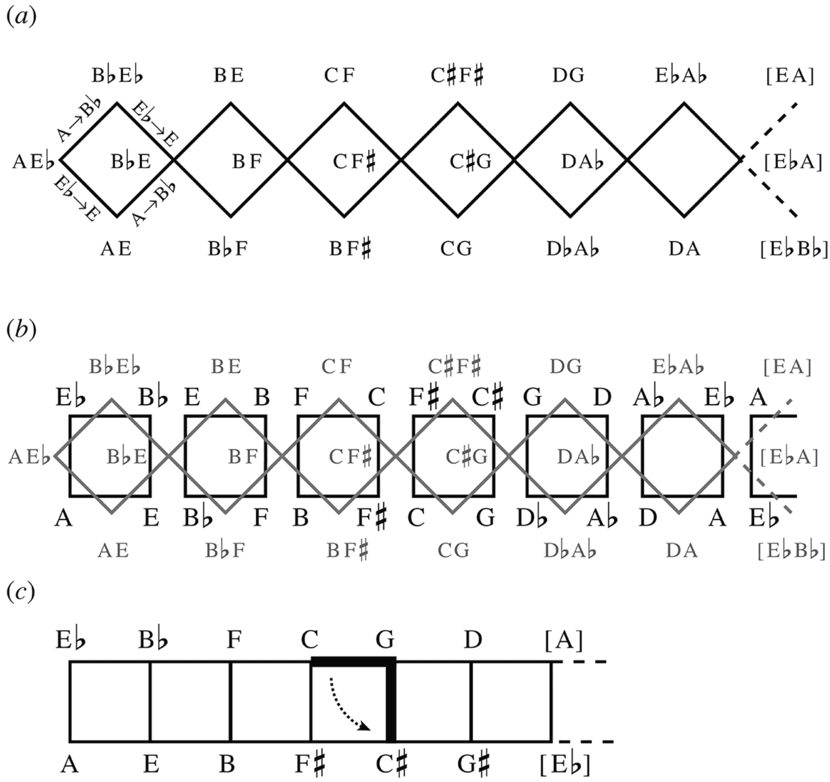


Figure 13. To form the note-based graph of nearly even two-note chromatic chords, we start with the chord-based graph at the center of two-note chromatic chord space (a) and then replace each cube with its dual (b). This produces a note-based graph (c) where line segments represent chords, and vertex-preserving "flips" represent single-semitone voice leading. Note that the rightmost square in panel c is linked to the leftmost square with a "twist," just as in panel a.

keeping B♭ fixed. Figure 13, b and c, constructs the dual graph, replacing each edge of the original with a point and connecting these new points by edges whenever the original edges met at a vertex. As discussed above, vertices in the new graph can be associated with pitch classes: each edge in the original graph is replaced by a point in the dual representing the note *that is not affected by motion along that edge*. What results is a series of disconnected squares in which horizontal edges represent perfect fourths while vertical edges represent tritones.

Note that the leftmost square in Figure 13b has B♭ above E, while the leftmost edge of the next square has E above B♭. Looking at Figure 13a, we can see why this is so: since every 45° line of the original space preserves a particular pitch class, B♭ is fixed by both the upper-right edge of the leftmost diamond and the lower-left edge of the diamond to its right; thus, in the dual squares, the pitch class B♭ is below E on one square, while the reverse is true on the neighboring square. To connect them, we therefore need to reflect every other square around its horizontal axis of symmetry, producing a circle of squares, each linked to its neighbors by shared edges.

In this dual graph, vertices represent notes, line segments represent chords, and efficient voice leading corresponds to "edge flips" around a common vertex. Observe, however, that the fourth C–G shares a vertex with *both* the tritone C♯–G and with the fourth G–D. To preserve voice-leading distances, we should consider the change from C–G to G–D to be a two-step motion, rather than a simple flip; otherwise, "flip distance" will not correspond in any obvious way to the total number of semitones moved by each voice.[21] I will say that the graph is *flip restricted*, since not all of its flips are size-one voice leadings. (In other words, if we want to model voice leading, we must *restrict* people from using these larger flips.) As we will see, flip restrictions often arise in complex note-based graphs.

First family, three dimensions and higher

Figure 14a shows the cubic lattice at the center of three-dimensional chord space. This structure was discovered by Douthett and Steinbach (1998) in a slightly more abstract form and is sometimes known as "Cube Dance."[22] As above, we can replace each cube by its dual octahedron, attaching them at their common faces to produce the lattice in Figure 14b.[23] (For clarity, Figure 15 shows how to construct the dual of an isolated cube in the original chord-based graph.) Efficient voice leading in the note-based graph is now represented by a "triangle flip" that links two triangles sharing a common edge. Note that a major triad such as C–E–G shares an edge with *both* the augmented triad C–E–G♯ and the minor triad C–E–A. (This is analogous to the way that C–G shared a vertex with C♯–G and G–D.) To accurately model voice leading, we must once again introduce flip restrictions, requiring that the major triad move first to the augmented triad before proceeding up to the minor triad.[24]

Let's now review the properties common to the two- and three-note cases. Figure 13c can be conceived as a circle of one-dimensional simplexes

21 If we consider the horizontal path from C–G to D–A to have length 2, then D♭–A♭ and D–A are equidistant from C–G. From a voice-leading perspective this is not true: D–A can pass through D♭–A♭ when it moves efficiently to C–G, but D♭–A♭ cannot pass through D–A when it moves efficiently to C–G. We lose the ability to model this when we consider horizontal flips to have unit length.

22 I will use the term "Cube Dance" in what follows, though I will typically use the term to refer to this graph-theoretical construction *as it is embedded in three-note chord space*. This embedding endows the lattice with more geometrical structure, allowing us to speak about "straight lines" that pass through the augmented triad. See Tymoczko 2011, chapter 3 and appendix C.

23 Douthett 2008 depicts one of these octahedra, dual to the individual cubes on "Cube Dance," and identified as the "hexatonic Tonnetz."

24 As noted above, allowing direct moves between these triads will have the consequence that "flip distance" no longer reflects voice-leading distance: F major will be just two flips from C major, whereas F minor will be three flips away. See Tymoczko 2010 and 2011, appendix C.

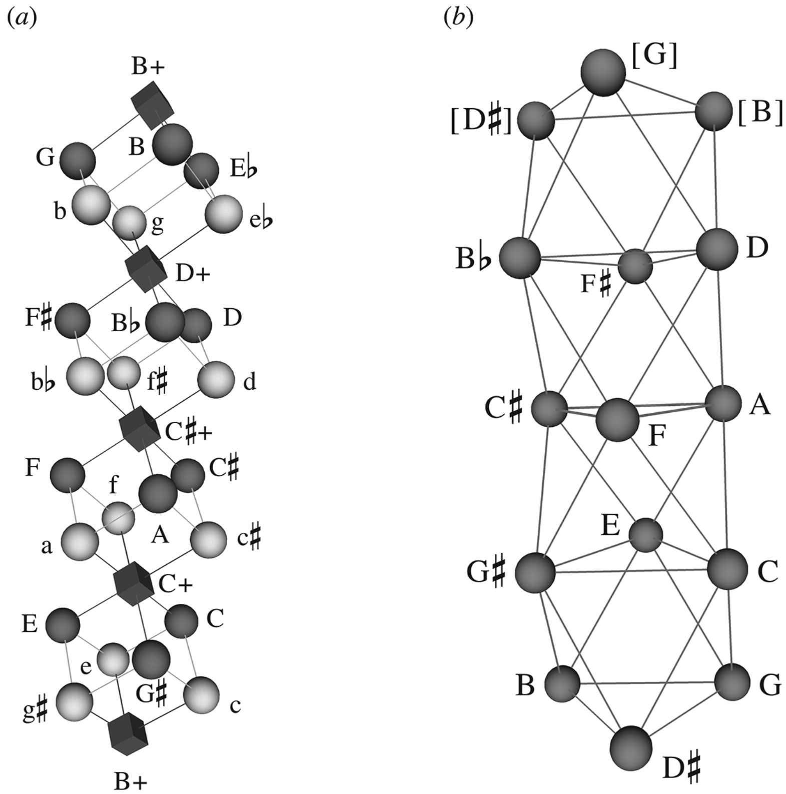


Figure 14. To form the note-based graph of nearly even three-note chromatic chords, we start with the chord-based graph at the center of three-note chromatic chord space (a); then we replace each cube with its dual and glue the resulting octahedra together in the appropriate way. This produces a circle of octahedra linked by common faces (b). Here, triangles represent major, minor, and augmented chords, and edge-preserving flips represent single-semitone voice leading. Note that the top face is a 120° rotation of the bottom face, indicating that the structure is globally twisted.

(vertical line segments) representing the completely even two-note chords (tritones). Each vertex in one tritone-simplex is connected by a line segment to all those notes in the neighboring tritone-simplexes *except for those that are a semitone away*. The tritone-simplexes thus form the "top" and "bottom" faces of a two-dimensional cross-polytope (= square, the dual of the two-dimensional cube, which is also a square).[25] Similarly, Figure 14b is a circle of two-dimensional simplexes (horizontal triangles) containing completely even three-note chords (augmented triads); each vertex in one augmented-simplex

25 Recall that every vertex of a cross-polytope is located either on the top face or on the bottom face; there are no vertices not contained by these two faces.

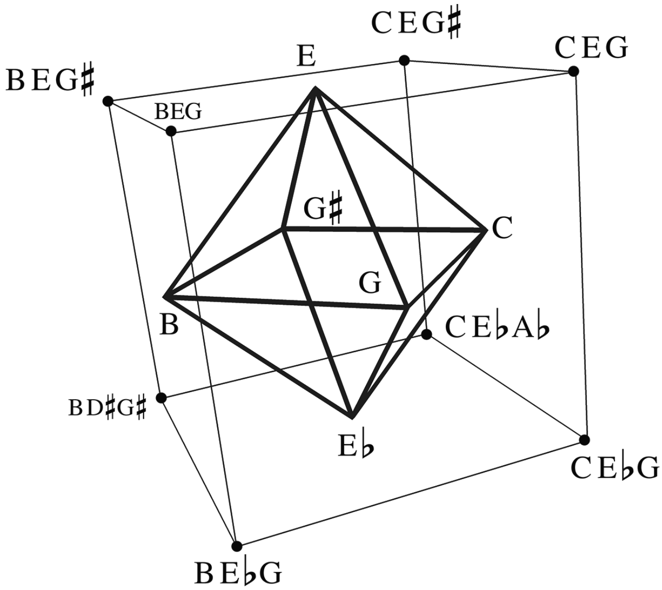


Figure 15. The duality between the chord-based cube and the note-based octahedron.

is connected to all the notes in the neighboring augmented-simplexes *except for those that are a semitone away*. The augmented-triad simplexes form the "top" and "bottom" faces of a three-dimensional cross-polytope (an octahedron, dual to the three-dimensional cube).

In four dimensions, we thus expect to find a circle of three-dimensional simplexes (tetrahedra) containing completely even four-note chords (diminished-seventh chords), with each vertex in one tetrahedron connected by line segments to all the notes in the neighboring tetrahedra *except for those that are a semitone away*. Somewhat surprisingly, it is possible to portray this figure in three dimensions. Figure 16 presents three nested tetrahedra, eliminating connections between them for the sake of visual clarity. (Figure 17 provides a glimpse of the chord-based dual, a circle of four-dimensional cubes, or "tesseracts," linked by shared vertices.) Since this is a three-dimensional representation of an inherently four-dimensional structure, it necessarily involves certain simplifications.[26] But it is clear enough to be useful as a model of tetrachordal voice leading.

Chords here are represented by tetrahedra. The completely even chords (diminished-seventh chords) are shown on the graph. To form dominant sev-

26 First, the inner (B diminished) tetrahedron should be understood to contain the outer (C♯ diminished) tetrahedron, since the graph is a "circle of tetrahedra." For the sake of clarity, I have not duplicated the B diminished tetrahedron. Second, the vertices of each tetrahedron poke through the faces of the tetrahedron enclosing it; this is simply for legibility—were each simplex entirely contained within the other, the smallest simplex would be much smaller than the largest.

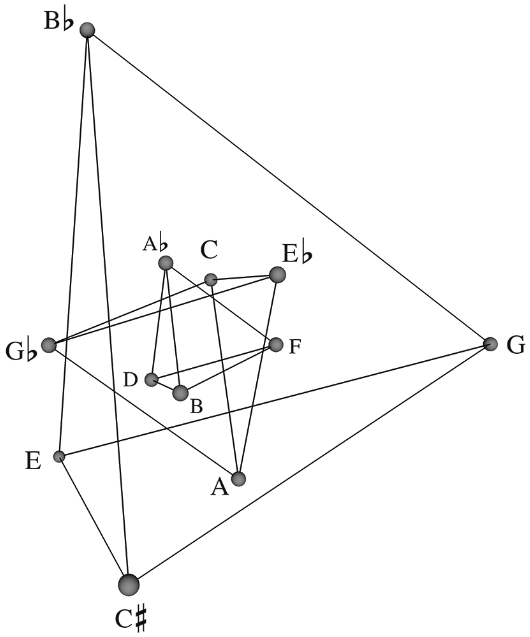


Figure 16. A note-based graph representing single-semitone voice leading among nearly even four-note chromatic chords. The graph is a series of nested tetrahedra, each dual to its neighbors. Chords are represented by tetrahedra drawing their vertices from (at most) two adjacent tetrahedra.

enths, minor sevenths, French sixths, and half-diminished sevenths, we combine vertices from adjacent tetrahedra, subject to the proviso that a tetrahedron cannot contain notes that are a semitone apart. (Semitonally related notes are maximally distant on adjacent tetrahedra and cannot be connected without cutting through one of the diminished-seventh chords; in this sense, they seem like they do not belong together.) Dominant and half-diminished sevenths contain a triangle from one tetrahedron and a vertex from another; this additional vertex is maximally close to the triangle in question. (For instance, the notes A♭ and B♭, on Figure 16's inner and outer tetrahedra, are directly above the triangle C–E♭–G♭ on the middle tetrahedron.) To form minor sevenths and French sixths, combine two edges from adjacent tetrahedra. For a given line segment, there is only one available line segment on another tetrahedron, and its position is visually obvious; for instance, A–E♭ on the middle tetrahedron can be combined either with C♯–G on the outer or B–F on the inner. (All other line segments on the inner and outer tetrahedra contain a note semitonally adjacent to either A or E♭.) Once again, semitonal voice leading is represented by a "simplex flip" that transforms a tetrahedron into another tetrahedron sharing a common face. The tetrahedron

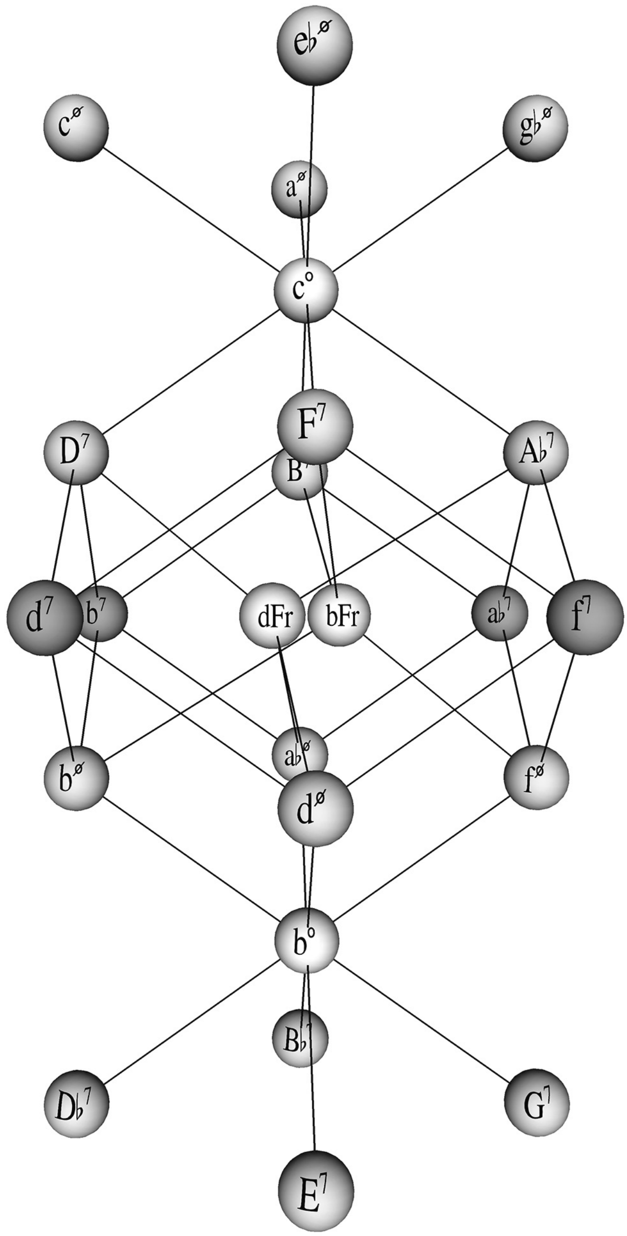


Figure 17. The chord-based graph at the center of four-note chromatic chord space. This represents the same musical possibilities as Figure 16.

C–E♭–G♭–B♭ shares a face with both C–E♭–G♭–A and C–E♭–G♭–A♭. To preserve voice-leading distances, we must treat the flip from $c^{ø7}$ to $A♭^7$ as being a size-two "compound" motion, for exactly the reasons discussed in the two- and three-dimensional cases.[27] Once again we encounter "flip restrictions," in this case to prevent us from flipping directly between half-diminished and dominant sevenths.

We can use the concept of "duality" to get a better grip on this fascinating figure. Each tetrahedron is drawn as the dual of its neighbors; for instance, the inner tetrahedron B–D–F–A♭ contains a vertex for each face of the tetrahedron C–E♭–G♭–A, and vice versa. Nearly even chords are combined by taking *dual elements from adjacent tetrahedra*: a face of one tetrahedron can combine with the dual vertex on its neighbor, forming dominant and half-diminished sevenths, just as an edge on one tetrahedron can be combined with the *dual edge* on a neighbor to form minor sevenths and French sixths.[28] The figure thus provides a very clear representation of common-tone relationships. For example, every face is dual to two vertices related by major-second transposition, just as every edge is dual to two edges related by that same interval. Thus, the face C–E♭–G♭ is dual to both A♭ and B♭ (forming $A♭^7$ and $c^{ø7}$), while the edge G♭–A is dual to both C♯–E and B–D (forming $f♯^7$ and b^7). We see that the progression (F♯, A, C♯, E) → (F♯, A, B, D) is in some sense analogous to (C, E♭, G♭, B♭) → (C, E♭, G♭, A♭), insofar as both combine a fixed, middle-simplex element (F♯–A or C–E♭–G♭) with both its outer-simplex and inner-simplex duals, related by major second in each case. Such relationships are much clearer in the note-based Figure 16 than in its chord-based counterpart.

At this point, the generalization to five and higher dimensions should be fairly clear: the chord-based graph is always a circle of hypercubes linked by shared vertices, while the note-based graph is always a circle of cross-polytopes linked by shared facets. (Alternatively, we can imagine a circle of (n –1)-dimensional simplexes, with each vertex connected to all the vertices of neighboring simplexes except those that are a semitone away.)[29] The simplicial facets of the cross-polytope represent chords, while single-step voice-

27 The graph in Figure 16 is a three-dimensional representation of an inherently four-dimensional structure, a fact that may lead some readers to wonder where the extra dimension has gone. The answer is, first, that the seventh chords (= tetrahedra) constructible on Figure 14 are not all congruent, while in four dimensions they are. (Note that diminished sevenths are all similar on Figure 14; I am talking here about minor sevenths, dominant sevenths, etc.) Furthermore, the radial dimension (outward from the graph's center) is doing double duty, with the interior of each diminished-seventh chord containing another diminished seventh. In the four-dimensional graph this is not the case, and the interior of every seventh chord is free of notes.

28 In a three-dimensional figure, let e be an edge between vertices A and B. The dual edge is the intersection of the faces $\mathcal{F}$ and $\mathcal{G}$, with $\mathcal{F}$ dual to A and $\mathcal{G}$ dual to B.

29 There is a slight complication in the case where there are only two simplexes, as with six-note chords in twelve-tone equal temperament. Here there are two separate sets of connections between the simplexes: going in one direction, each vertex is connected to all the neighboring simplex's vertices *except for the one that is a scale step above it*, while in the other direction each vertex is connected to all the neighboring simplex's vertices *except for the one that is a scale step below it*. This six-dimensional figure is graph-theoretically identical to Walter O'Connell's "tone lattices" (1968).

leading corresponds to “flips” that preserve one of the simplex’s own facets (ridges of the original cross-polytope). As in the previous cases, some simplexes will share facets both with a perfectly even chord and with one of their own inversions; to model voice-leading distances, we must require that they move *first* to the perfectly even chord before proceeding onward to their inversions.

Second family, one chord type

We now turn to the second family of chord-based graphs, arising when the size of the chord is relatively prime to the size of the scale. As mentioned above, the dimension of these graphs is controlled not by the number of notes in the chord but rather by the *number of chord-types we wish to represent*. Thus, if we are concerned only with maximally even chords, we can produce a one-dimensional voice-leading graph, no matter how large the chord happens to be. The second dimension is needed when we also want to represent the second most even type of chord, just as we need three dimensions to represent the third most even type of chord, and so on.

When we restrict our attention to just one chord type—the maximally even chord—things are attractively simple. The chord-based graph is a “generalized circle of fifths” that links transpositionally related chords by single-step voice leading (see Figure 10). To form the note-based dual, replace every vertex on the chord-based graph with an $(n - 1)$-dimensional simplex (n being the number of notes in the chord, and the number of vertices in the simplex). The result is a “circle of simplexes” each linked to its neighbors by a shared facet. Figure 18 illustrates the one- and two-dimensional cases. Since we require a vertex for each note, the note-based graphs involve progressively more and more dimensions. Indeed, the note-based analogue to the circle of fifth-related diatonic scales (the true “circle of fifths,” whose one-dimensional chord-based graph is shown in Figure 19) would be six-dimensional!

The triadic case has previously been explored by Candace Brower (2008), who observes that the rightmost triangle on Figure 18b needs to be attached to the leftmost with a “twist.” Were we to embed this graph in two dimensions, it would therefore lie on a Möbius strip. (Alternatively, we can imagine embedding this figure in three-dimensional chord space, where the “twist” is partially supplied by the space itself.) Another way to think about the construction is that it is a common-tone Tonnetz for an inversionally symmetrical triad. Figure 20 shows that for a “generic” trichord, there are three separate inversions that preserve two of the chord’s notes. For an inversionally symmetrical trichord, however, there are only two such inversions, since one inversion reproduces the original trichord. Since the diatonic triad is inversionally symmetrical, its “common-tone Tonnetz” must therefore consist of a single strip of triangles, in contradistinction to the planar Tonnetz we are all familiar with.

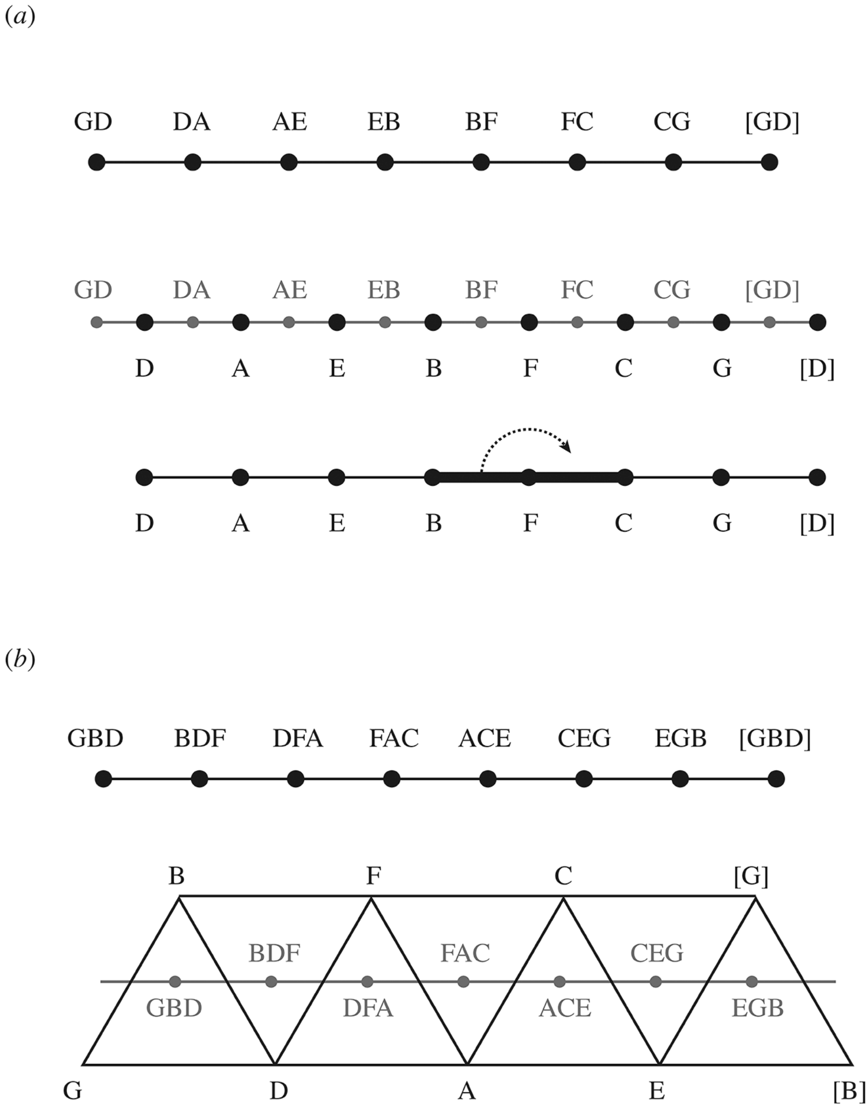


Figure 18. For an *n*-note chord, the "generalized circle of fifths" is represented by a one-dimensional chord-based graph and an (*n* – 1)-dimensional note-based graph. Here we construct the note-based graphs representing stepwise voice leading among maximally even two- and three-note diatonic chords, shown in (a) and (b), respectively.

Figure 21 shows the four-note analogue to Brower's graph, a chain of tetrahedra (three-dimensional simplexes) representing diatonic seventh chords. The circle of thirds spirals around the exterior of this structure in a way that is distantly reminiscent of Elaine Chew's (2000) "spiral array." Since the "spiral of thirds" takes approximately a quarter-turn with each step, one can find vaguely straight "lines of seconds" on the figure, for instance,

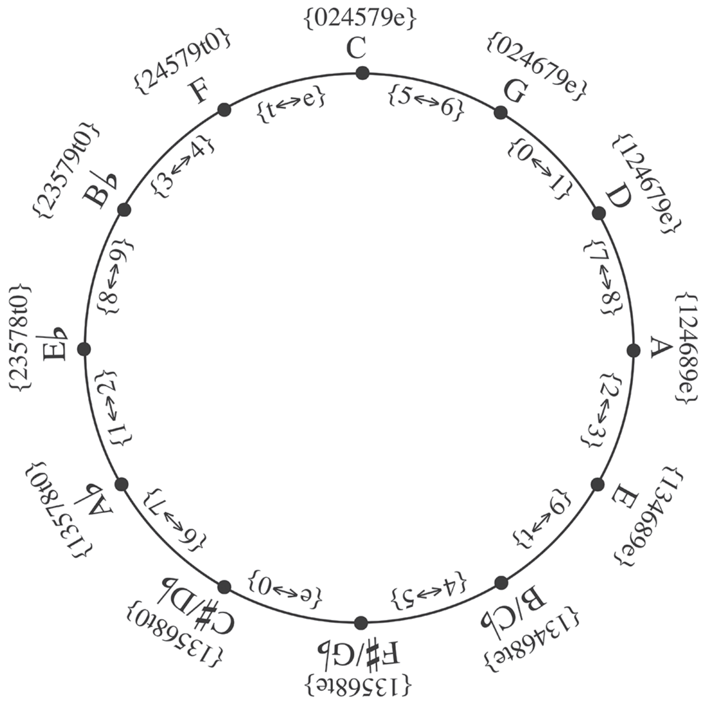
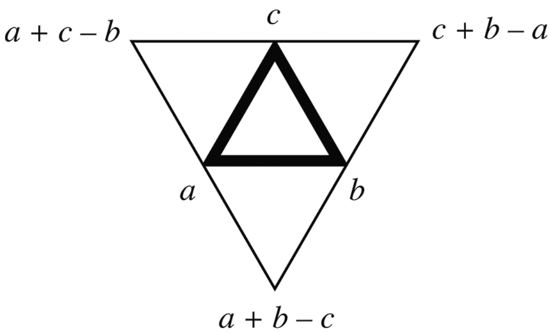


Figure 19. The traditional "circle of fifths" represents single-semitone voice leading among seven-note diatonic scales. The chord-based graph is one-dimensional, while its note-based analogue requires six dimensions.

C–B–A–G at the top, E–D–C–B at the bottom front, and G–F–E–D at the bottom rear.

Thus, where our first family of note-based graphs (Figures 13, 14, and 16) contains "generalized octahedra" (cross-polytopes) linked by shared facets, the second family—at least in its most elementary manifestations—contains "generalized tetrahedra" (simplexes) linked by shared facets (Figures 18 and 21). With the first family, the octahedra are linked by simplicial faces representing completely even chords. Since these chords share no notes with their semitonal transpositions, we are required to form "hybrid" chords combining the notes of adjacent simplexes—yielding "nearly even" chords such as perfect fifths, major and minor triads, and dominant seventh chords, all of which combine the notes of two semitonally adjacent perfectly even chords.[30] With the second family, the maximally even chords are not *completely* even, and neighboring chords turn out to share *all but one* of their notes. This allows us to

30 Each of these chords has a partner that can be connected to it by what Robert Cook (2005) calls "extravagant" voice leading, in which every note moves semitonally (e.g., C major and A♭ minor, C^7 and $e\flat^{ø7}$, C♯–D♯–E–G–A♭–B♭ and C–D–F–F♯–A–B, which are not inversionally related). However, there is also an alternative generalization of Cook's "extravagance" in terms of near inversional symmetry, as in (C, D♯, E) → (D♭, D, F).



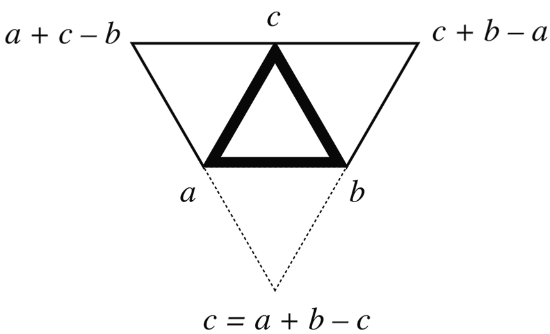


Figure 20. (a) A generic triad shares two notes with three of its inversions. For instance, if we start with the C major chord (*a* = 0, *b* = 4, and *c* = 7), then the bottom vertex is A, the leftmost vertex is E♭, the rightmost vertex is B, and the three peripheral triangles represent A minor, C minor, and E minor. (b) By contrast, if we start with an inversionally symmetrical triad, then one flip reproduces the original chord. For instance, suppose we start with the C diatonic triad (*a* = 0, *b* = 4, *c* = 2, measuring in diatonic scale steps). Then *c* = *a* + *b* – *c*, and the triad is connected to only two distinct inversions.

form a "circle of simplexes" sharing a common facet, leaving no possibility for the "hybrid chords" in the first family of lattices.

Second family, two or more chord types

We now turn to figures that represent more than just maximally even chords. Figure 22 uses the graph of single-step voice leading among diatonic thirds and fourths to generate the note-based analogue. What is surprising is that the note-based graph contains redundancies, multiple line segments representing the same chord; for instance, the tritone B–F appears on three con-

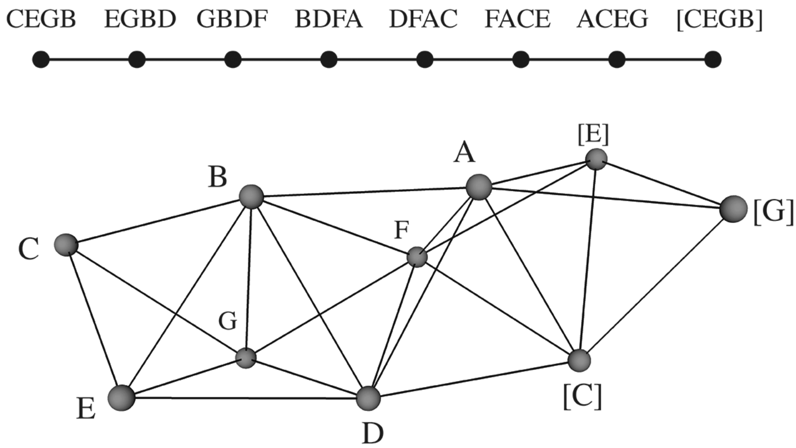


Figure 21. Voice-leading relations among maximally even diatonic seventh chords are represented by a circle of tetrahedra, each linked to its neighbors by a shared face.

secutive line segments located on the three leftmost squares. This is because the original, chord-based graph contains squares that are joined at a common face: when we take the dual of each of the graph's component squares, the shared vertices in the chord-based graph turn into duplicated edges in the note-based graph.

These redundancies are unattractive, and one's initial instinct is that there must be some simple way to remove them. But this turns out to be easier said than done. Figure 23b removes some—but not all—of the duplications, transforming two adjacent squares into a single square with a point at its center (C–F still appears in the two leftmost squares). However, this process has created edge flips that represent nonstepwise voice leading; for instance, F–A and F–D share a vertex on the new graph, even though the voice leading (F, A) → (F, D) moves one voice by *three* steps. By contrast, *all* the edge flips on Figure 23a represented single-step voice leadings, as do all edge flips on the standard Tonnetz. To use our new graph to represent voice-leading distances, we must therefore introduce "flip restrictions," disallowing nonstepwise flips, much as we disallowed the direct move from C–G to G–D on Figure 13.

Thinking about it a little more, it becomes clear that redundancies and flip restrictions are not unique to this second graph family. Consider that when we took the dual of the squares in Figure 13a, we created *a series of disjoint squares in which every tritone was redundantly represented* (Figure 13b). We were able to remove these redundancies by gluing together adjacent squares, as in Figure 13c, but only at the cost of introducing flip restrictions, as when we declared that C–G could not directly flip to G–D. (There was no possibility of such flips when we considered a series of disjoint cubes.) Figure 24 describes another case in which redundancies appear, extending our earlier

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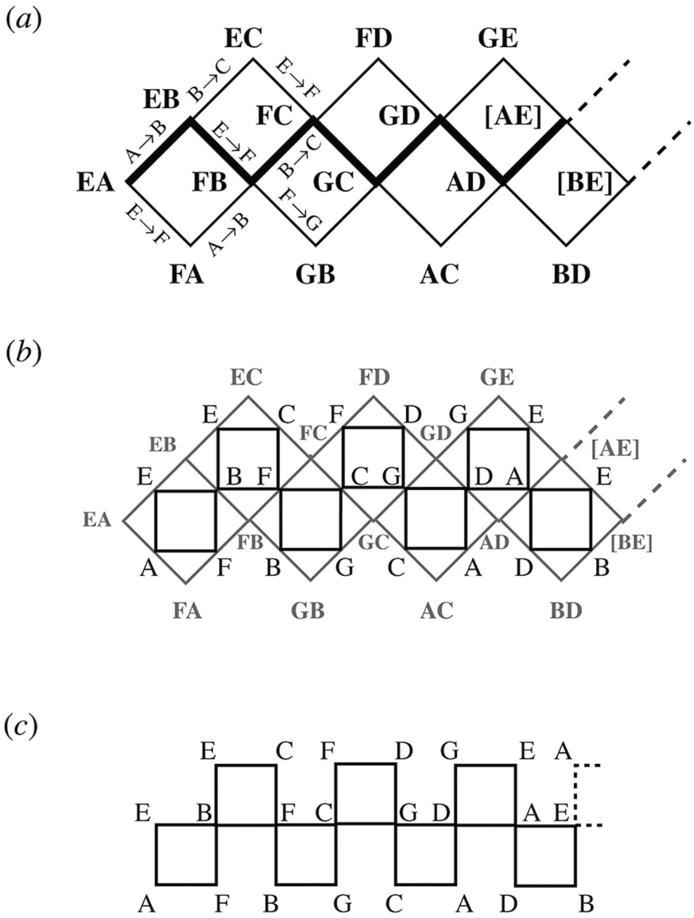


Figure 22. Constructing the dual lattice representing voice leading among the two most even kinds of two-note diatonic chord (i.e., thirds and fourths). In (a) we begin with the chord-based lattice. We then take the dual of each cube (b), producing a redundant lattice in which some chords appear multiply.

graph of two-note chords (Figure 13a) so as to include major thirds. This introduces shared faces into the chord-based graph, which in turn produce redundancies in the note-based analogue. From this perspective, the two families of lattice are not fundamentally dissimilar: the only difference is exactly where the redundancies (or flip restrictions) happen to appear.

In fact, we can make this point more precisely. Suppose that a chord-based voice-leading graph contains a point where a single note can move either upward or downward by step: symbolically, we can write $A \leftarrow B \rightarrow C$, meaning that at chord B a note can move either down by step to form chord A or up by step to form chord C. In the note-based analogue, single-step voice

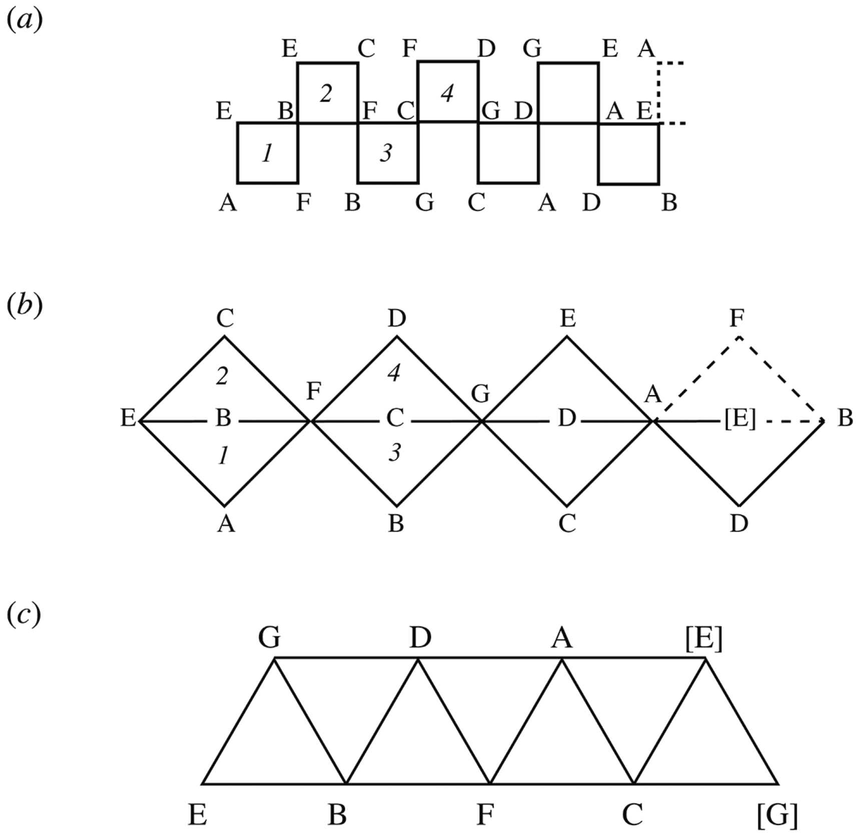


Figure 23. We can eliminate some duplications by combining adjacent cubes in the chord-based graph. Here, for instance, squares 1 and 2 in (a) become the leftmost square-with-a-central-point in (b). As a result, some "flips," such as (F, A) → (F, D), will represent non-stepwise voice leading. (c) We can also use Brower's graph of diatonic triads to represent voice leading among diatonic thirds and fourths.

leadings will be represented by "flips" that connect simplexes sharing a common facet. It follows that either (a) *A*, *B*, and *C* will all share a common facet, and the chord-based graph will be flip restricted;[31] or (b) *B* will appear redundantly on the graph. Thus, redundancies and flip restrictions, rather than being problems to be avoided, are actually intrinsic to complex note-based graphs.[32] What is remarkable, perhaps, is that the standard Tonnetz contains

31 *A* and *C* share a common facet but by hypothesis are not connected by single-step voice leading: since single-step voice leading (in the same voice and in the same direction) takes *A* to *B* and *B* to *C*, the voice leading $A \rightarrow C$ moves one voice by *two* steps.

32 Another way of thinking about the underlying issue is that there are fundamental divergences between *common-tone* and *voice-leading* distances. From a common-tone perspective, F–A is just as close to F–D as it is to F–B, since they all share the note F. But from a voice-leading perspective, F–A is *closer* to F–B than it is to F–D. Redundancies serve the function of preserving voice-leading distances in the face of shared common tones. To collapse these duplications is to begin to prioritize common-tone distance over voice leading.

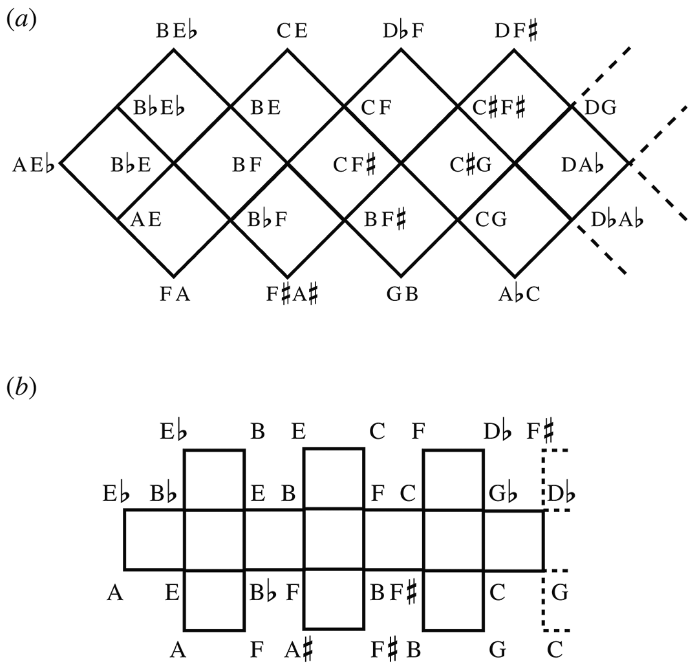


Figure 24. Redundancies can occur in the first family of lattices as well. Here, (a), we start with a graph containing the three most even types of two-note chromatic chords (i.e., tritones, perfect fourths, and major thirds). Since some squares are linked by a common face, the dual contains duplications (b).

no such redundancies or flip restrictions. This is because we can use "parsimonious" (or single-step) voice leading to either raise or lower each note of every triad, but not both.[33]

Somewhat surprisingly, however, we can use Brower's *three-note* "diatonic Tonnetz" (Figure 18b, reproduced here as Figure 23c) to represent voice-leading relations among nearly even *two-note* diatonic chords. Brower's graph is redundancy free, with every diatonic third and fourth appearing as a line segment in exactly one place on the graph. (Indeed, it has no superfluous connections or line segments, containing exactly the edges that are needed for this particular purpose.) From a contrapuntal perspective, however, the figure is a bit perverse, as stepwise voice leading is reflected by *maximally inefficient* flips defining internal angles greater than 90°.[34] Flips such as (F, A) → (F, D) are *visually* more salient, connecting two edges of a triangle (and spanning only 60°), but represent musically *inefficient* voice leadings in which a voice moves by third or fourth. This inverse relation

33 For instance, the C and E of C–E–A can be raised but not lowered, while the A can be lowered but not raised.

34 For example, the voice leading (F, A) → (F, B) is reflected by a 120° flip, while (F, B) → (F, C) spans 180°.

between geometrical and contrapuntal distance is problematic insofar as the central goal of geometrical music theory is to construct spaces in which geometrical proximity models musical proximity.

Once again, these ideas can be generalized to higher dimensions. Figure 25a shows a three-dimensional chord-based graph representing efficient voice leading among diatonic triads, fourth chords, and incomplete sevenths; Figure 25b shows the note-based analogue, a circle of octahedra linked by shared vertices.[35] Again, we encounter redundancies, with the F major triad being represented by three separate triangles on Figure 25. Figure 26a removes some of these duplications exactly as before, by combining two adjacent octahedra into a single octahedron with a point at its center. This transformation comes at the cost of introducing flip restrictions, since some edge flips (such as C–G–A → C–B–A, which share a face) now represent nonstepwise voice leading. Alternatively, and again somewhat surprisingly, we can use the three-dimensional, tetrahedral Tonnetz (Figure 26b) to model diatonic trichords, with chords being represented either as triangles (e.g., E–G–B) or as open line segments (e.g., C–G–D). Since this graph is completely redundancy free, it requires a number of flip restrictions. Interested readers are invited to investigate further.

The appearance of redundancies and flip restrictions is disappointing, in large part because the familiar Tonnetz has conditioned us to expect graphs without these features. One might have hoped that there was an elegant family of redundancy-free, Tonnetz-like graphs for whatever musical situation we might happen to find ourselves in. Instead, however, it seems that flip restrictions and redundancies are inherent in the very project of creating generalized note-based graphs, avoidable in just a few special cases. Even modest extensions to the Tonnetz, such as the "chain of octahedra" in Figure 13, require flip restrictions. It is ironic that the first and earliest example of a note-based graph would turn out to be such an unusual case.

❋ ❋ ❋

Though this section has constructed just a few lattices from just two families of graphs, our procedures are applicable more broadly: virtually any sufficiently complete chord-based voice-leading lattice will be composed of (hyper)-cubes, and these can always be converted via duality to cross-polytopes in which notes represent chords (see Section 2); it is just a matter of determining how these cross-polytopes intersect with one another and of locating any duplications they might contain. Thus, very little of our work depends on the particular structure of the diatonic and chromatic collections: the important

35 Note the inverse relationship between note-based and chord-based graphs: if the hypercubes in the chord-based graph intersect at common *vertices*, then the note-based cross-polytopes will intersect in shared *facets*; conversely, if the chord-based cubes intersect in shared *facets*, then the note-based cross-polytopes will intersect in shared *vertices*. This, of course, is a consequence of the way duality exchanges vertices and facets.

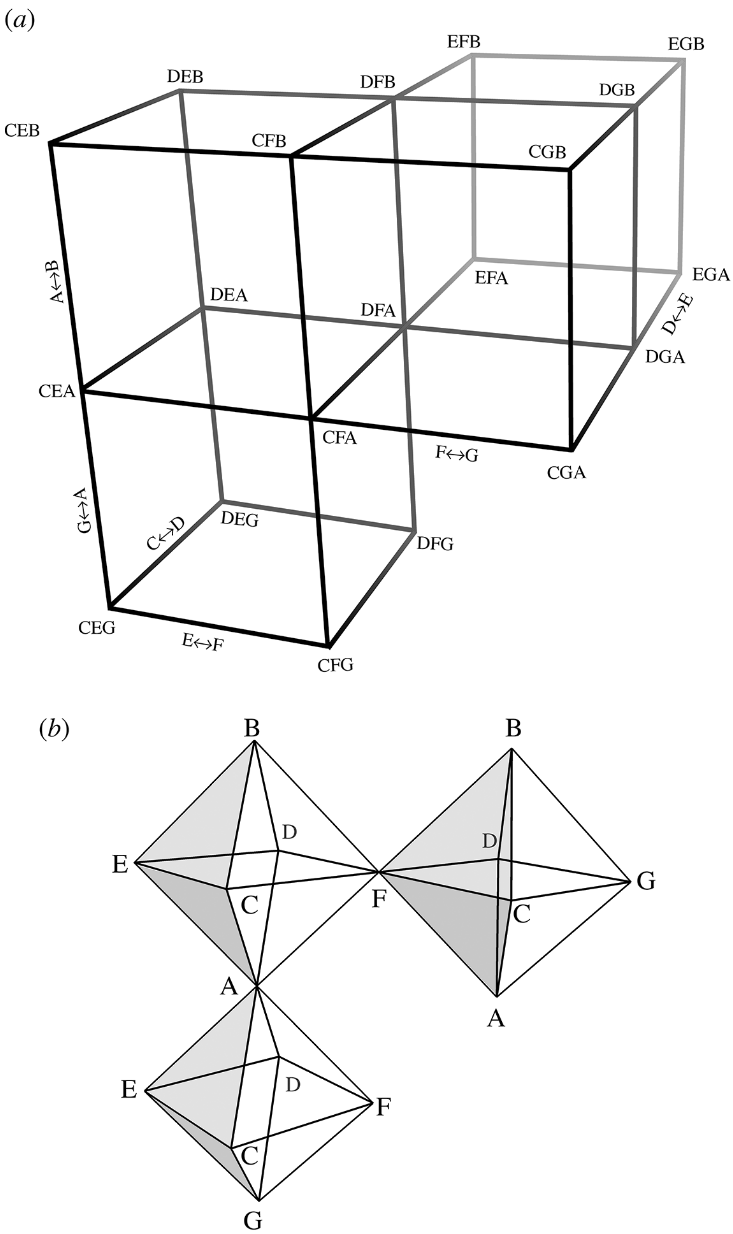


Figure 25. **(a) The chord-based graph representing voice-leading relations among the three most even types of three-note diatonic chords, a circle of cubes linked by shared faces. (b) Its note-based version, a circle of octahedra linked by shared vertices. Note that the top graph contains four cubes, whereas the bottom graph, for clarity, contains only three octahedra.**

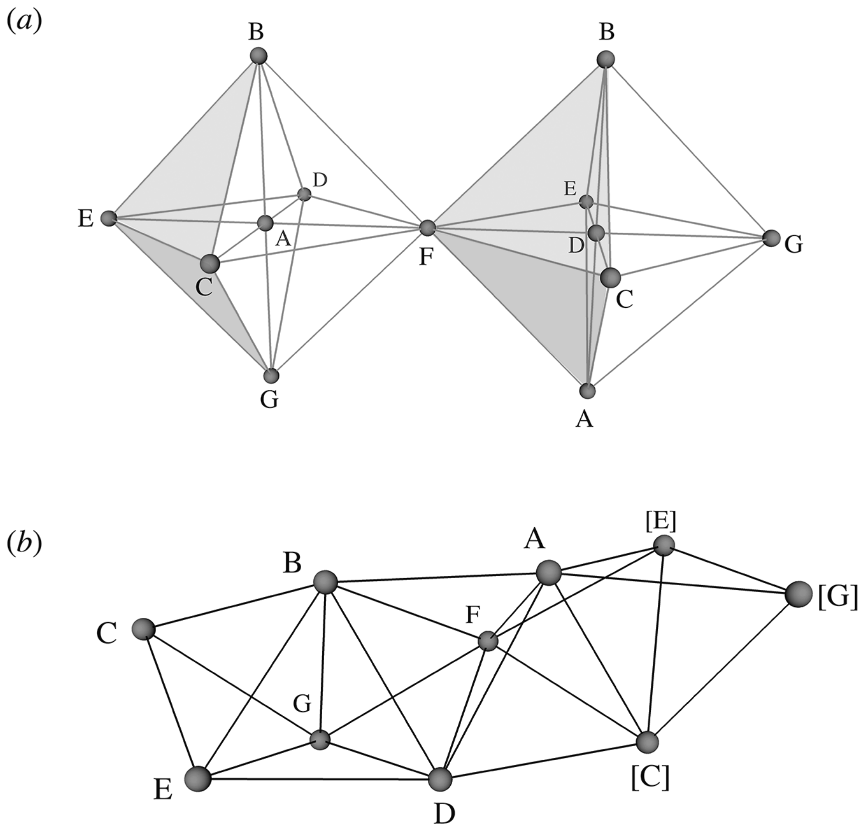


Figure 26. (a) We can glue together adjacent octahedra in Figure 25, forming octahedra with central points. Once again, however, the resulting graph will have "flips" that represent nonstepwise voice leading—including (C, G, A) → (C, B, A). (b) We can also represent voice leading among diatonic trichords using our "circle of tetrahedra." Here, trichords are represented by triangles (e.g., E–B–D) or chains of line segments (e.g., C–G–D). In either case, we need flip restrictions.

variable is simply the relative size of chord and scale.[36] The one exception concerns the use of higher-dimensional diatonic models to model smaller diatonic chords, as in Figures 23c and 26b.[37]

4. Generalizing the Tonnetz Proper

So far, we have come close to the original Tonnetz without recreating it exactly: the "circle of octahedra" shown in Figure 14 is a three-dimensional

36 In particular, the preceding ideas are straightforwardly applicable to the hexatonic, octatonic, whole-tone, melodic minor, and many other scales. See Tymoczko 2011, chapter 4.

37 There is no guarantee that we could always use a model of trichords to represent voice leading among intervals (Figure 23c) or a model of tetrachords to represent trichords (Figure 26b). Interested readers are invited to consider the conditions under which these constructions can be generalized.

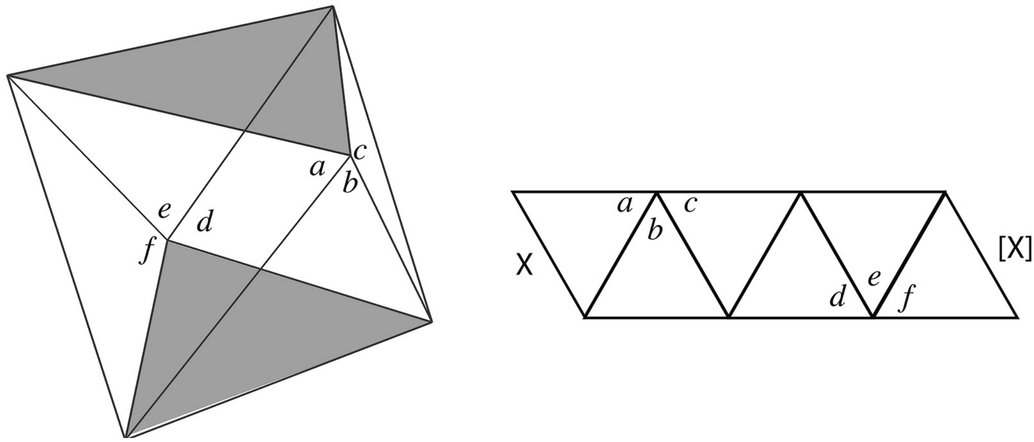


Figure 27. If we remove two opposite faces of an octahedron (left), we can unfold the remaining faces into a "circle of triangles" (right), shown here as a chain whose right edge is the same as its left.

figure that displays single-semitone voice leading among major, minor, *and augmented* triads, whereas the standard Tonnetz shows only major and minor chords. The question, then, is how to eliminate the augmented triad so as to recover the Tonnetz proper.

The trick—and it is a subtle one—is to forget the augmented triad by reconceiving our graph's topology. Essentially, we declare that *the triangle representing the augmented triad, rather than enclosing a region of three-dimensional space, is actually a circular dimension unto itself.* (That is, we stop conceiving of the augmented triangles extrinsically, as embedded in a surrounding three-dimensional space, and start thinking of them intrinsically, as one-dimensional spaces unto themselves, topologically equivalent to the circle.)[38] This has the effect of converting our structure from a three-dimensional lattice, embedded within twisted three-dimensional space, into a two-dimensional lattice living on a two-dimensional torus. It also has the effect of creating a very sharp distinction between the augmented triads and the major and minor triads. Figure 27 represents this visually: we declare that the top and bottom triangles of each octahedron are dimensions unto themselves, unlike the space-enclosing triangles comprising the rest of the figure. This allows us to inscribe the remaining six triangles onto a cylinder. (These triangles together comprise the "LP cycle" of semitonally related hexatonic triads.) Unrolling the entire stack of octahedra produces the two-dimensional Tonnetz shown in Figure 1a.

38 As promised in Section 1, intrinsic and extrinsic are involved in a subtle dance. Essentially, the music-theoretical question is whether to consider the augmented chord as a triangle or as a circular dimension.

This reconstruction of the Tonnetz sheds new light on an issue that is currently the subject of spirited theoretical debate. Recently, Richard Cohn responded to criticism that the Tonnetz does not faithfully represent voice-leading distances by proposing to include augmented triads on the structure.[39] On Cohn's revised Tonnetz, the self-intersecting line segment C–E–G♯–C is to be counted as triangle just like C–E–G–C and A–C–E–A. Cohn further declares that the triangle C–E–G–C cannot be flipped directly onto A–C–E–A but must first pass through C–E–G♯–C. (This flip restriction has the effect of converting the "R voice leading" into a size-two move, consistent with the fact that it moves its voices by two total semitones.) When I first encountered Cohn's proposal, it struck me as fairly ad hoc, largely because the "triangle" C–E–G♯–C is geometrically very different from those representing major and minor triads: the former is a circular dimension unto itself, enclosing no surface area on the toroidal Tonnetz, while the latter is a generic triangle within the space. (Indeed, the entire mathematical subject of *simplicial homology* centers around this distinction.) But once we reconceive the Tonnetz as a *three-dimensional* structure, dual to the stack of cubes at the center of three-note chord space, Cohn's construction looks considerably more natural: on the three-dimensional version of the Tonnetz shown in Figure 14, the augmented triad is no less triangular than the other chords. In this sense, the three-dimensional Tonnetz is the natural geometrical environment for Cohn's current work.[40]

This should lead us to ask whether the (equal-tempered) Tonnetz is in fact truly toroidal. Previous theorists have unanimously answered this question affirmatively, to the point where one would almost court ridicule to suggest otherwise.[41] But our discussion has given us reason to be more circumspect. Considered *as a graph*, the Tonnetz is simply a collection of vertices and edges having no particular geometry or topology. To embed this graph into a robustly geometrical space requires us to ask questions like "should the 'major third axis' be a single straight line?" or "should the edges representing augmented triads be similar to those representing major and minor triads?" Our answers, rather than being simple consequences of the Tonnetz's graph-theoretical structure, will depend on what we want to do with the space.[42]

39 See Cohn 2011b. For the original criticism, see Tymoczko 2009a, 2010, and 2011.

40 Cohn, however, continues to use the traditional, two-dimensional Tonnetz, which in my view draws an unnecessarily sharp distinction between the augmented and the other triads.

41 For claims that the Tonnetz is toroidal, see Cohn 1997 and Gollin 1998, among many other examples. As far as I know, no previous theorist has ever considered the possibility that the equal-tempered Tonnetz could be anything other than a torus.

42 The assumption that the Tonnetz is toroidal is an example of music theorists implicitly attributing structure to their models over and above that contained in their mathematical formalism. For other examples, see Tymoczko 2008a and 2009b, which argue that David Lewin implicitly conceived of group elements as having magnitudes. Indeed, Lewinian formalism, being much closer to graph theory than to geometry, may have helped to obscure the sorts of questions we have been asking.

And here it becomes vitally important that Tonnetz is a multivalent structure. From a purely acoustical point of view, it is eminently reasonable to arrange the notes of the augmented triad along a straight line that returns to itself, creating an "axis of thirds" and an "axis of fifths." (Thus, when modeling acoustics, we are implicitly assigning to the Tonnetz more than graph-theoretical structure, asserting that it has "straight lines" rather than simply connections among vertices.) But insofar as the Tonnetz is understood as a model of voice leading, these acoustical desiderata take a back seat to the goal of faithfully representing contrapuntal distances. It follows that we should represent the voice-leading Tonnetz as a *nontoroidal, three-dimensional structure whose individual octahedra are the duals of the cubes in "Cube Dance," and in which major, minor, and augmented triads are all on an equal footing*. In this three-dimensional structure, we no longer have linear "axes" representing motion by major third, minor third, or perfect fifth, having sacrificed this acoustical nicety in order to make room for the augmented triads.[43] To the graph theorist, the two structures the same, but to the topologist, geometer, or music theorist, they are quite different. If this seems surprising, it is only because we have been conditioned to assume that there is a single, univocal Tonnetz that can represent *both* acoustics and voice leading—and perhaps even common-tone retention as well.

Having examined the three-note Tonnetz, let us now turn to its four-dimensional analogue. Since the Tonnetz eliminates augmented chords, directly connecting major and minor triads by way of the "R voice leading," we expect that the four-dimensional graph will eliminate diminished sevenths in favor of direct connections between chords such as C^7 and $e^{ø7}$, which share the diminished triad E–G–B♭.[44] And just as we formed the three-note Tonnetz by pulling apart the augmented triad, so that the two-dimensional triangle becomes a one-dimensional circle, we will form the four-note Tonnetz by flattening three-dimensional tetrahedra (Figure 16) into two spherical dimensions. Figure 28 shows two ways to flatten a tetrahedron: in the first, one vertex appears in the center of a triangle, while in the second, it appears in three separate places, lying beyond the triangle's vertices. The four-note Tonnetz, shown in Figure 29, can thus be represented as a series of layers, each identical to one of these two-dimensional representations. Notes on one layer are connected by edges to all notes on the adjacent layers except those that are a semitone away.[45] (Again, I leave out the cross-layer connections for the sake of visual clarity.) Chords are represented by (space-enclosing) tetrahedra that draw their notes from two adjacent layers.

43 In Figure 14b, augmented triads are equilateral triangles rather than straight lines. Similarly, the paths representing perfect fifths and minor thirds change direction at every vertex.

44 Douthett and Steinbach (1998) call this voice leading "R*."

45 Note that the vertex outside the triangle in one layer is not connected to the vertex at the center of the triangle in the next layer. Note also that the triangle in one layer is dual to those in the adjacent layer: a vertex on one triangle is not connected to the vertex that is the dual of the original vertex's opposite edge.

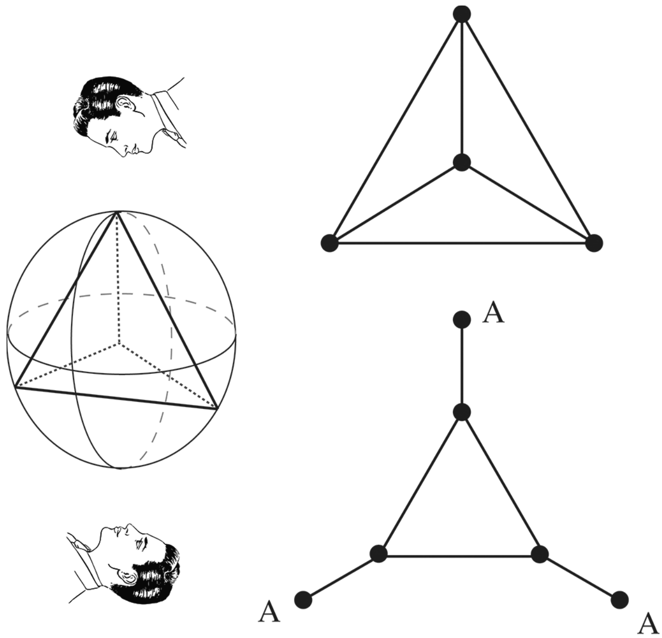


Figure 28. Two ways of flattening a tetrahedron. If we look from above, the fourth vertex appears to be in the center of the triangle. From below, the fourth vertex is reachable in three separate directions. (To see this, imagine cutting the globe at the north pole and spreading it flat. No matter which direction we go, from the perspective of the south pole, we will eventually get to the north pole.)

One interesting feature of the four-note Tonnetz (and indeed, the analogous constructions in all dimensions) is that it is *homogeneous*: at every point in the space, each note can be moved stepwise up or down, but not both. (This homogeneity is precisely what allows us to avoid flip restrictions and redundancies, as discussed in Section 3.) For instance, the root of the dominant seventh can be raised by whole step to produce a half-diminished seventh, while the third, fifth, and seventh can each be lowered by half step, producing a minor seventh, a French sixth, and a minor seventh, respectively.[46] Figure 30 tries to illustrate this by representing only those connections that participate in single-step voice leading from the C half-diminished chord. The figure consists of four "peripheral" tetrahedra surrounding the central C half-diminished tetrahedron, with each peripheral tetrahedron sharing three notes (and hence a face) with the original. Since the graph is homogeneous, the neighborhood around *any* tetrahedron will look locally like the one

46 By contrast, the lattice at the center of four-note chord space is not homogeneous, since each note of the diminished seventh can be raised *or lowered* by semitone; in eliminating the perfectly even chord, we remove this inhomogeneity, which in turn eliminates the need for redundancies and flip restrictions.

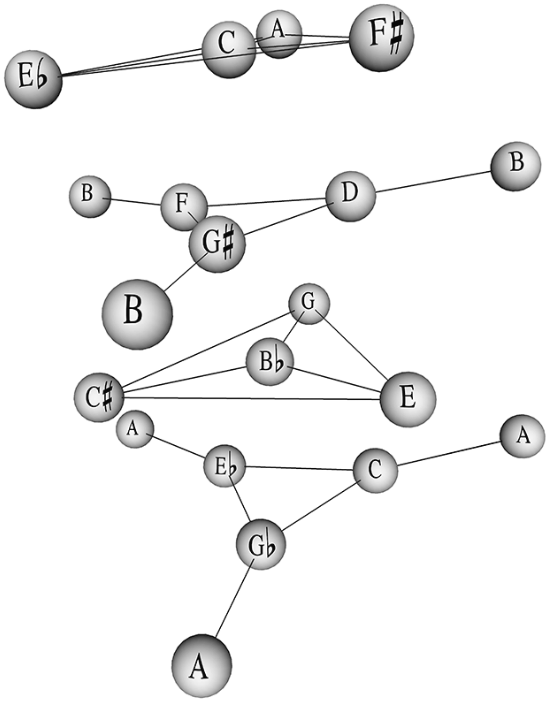


Figure 29. The four-note Tonnetz, with the diminished sevenths flattened into two dimensions. Each note on one layer can be connected to all the notes on the adjacent layers, except those a semitone away. If we require that chords be three-dimensional figures, then the diminished-seventh chord is no longer available, since all its notes lie in two dimensions.

shown in the figure. Thus, by relabeling the vertices on our graph, we could represent the voice-leading possibilities for *any* chord in the four-note Tonnetz (Figure 31).

The four-note Tonnetz is complex enough to justify a look at its chord-based dual, whose cross sections are shown in Figures 32 and 33. (This structure is to our four-note Tonnetz as the "chicken-wire torus" is to the original three-dimensional Tonnetz [Figure 1].) The graph is a circle of "rhombic dodecahedra," analogues to the hexagons shown in Figure 1b.[47] Each rhombic dodecahedron contains four dominant seventh, minor-seventh, and half-

47 The hexagon is the shape we get when we project a three-dimensional cube into two dimensions, along the diagonal line connecting two opposite vertices. (For instance, we can project the cube whose coordinates are all ±1 into the plane whose coordinates sum to 0.) The rhombic dodecahedron is the shape that results from the analogous projection of a four-dimensional cube. Musically, these projections arise when we remove the augmented triad from the cubic lattice at the center of three-note chord space, or the diminished-seventh chord at the center of four-note chord space.

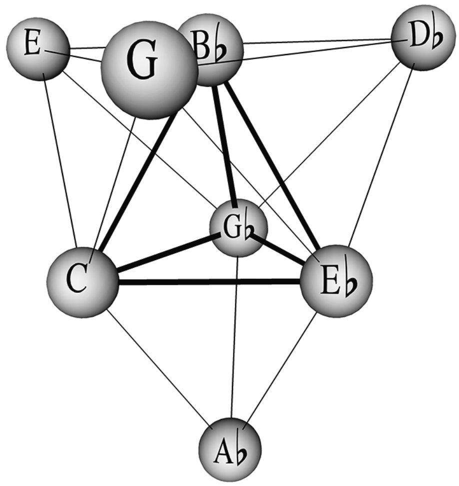


Figure 30. The tetrahedron representing C half-diminished shares a face with four other tetrahedra. As a result, there are four possible "simplex flips," each of which raises or lowers a different note of the original chord.

diminished seventh chords, as well as two French sixths; all these chords draw their notes from two adjacent diminished sevenths. The dominant seventh chords on one dodecahedron are linked to the half-diminished sevenths on another dodecahedron by single-step voice leading.[48]

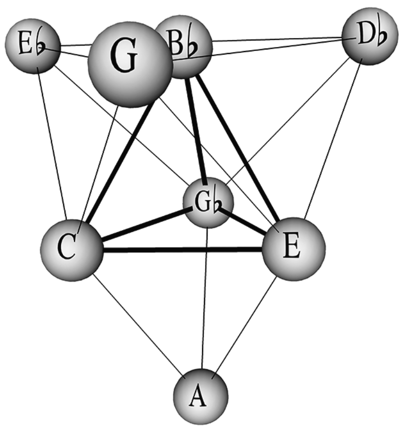
Interestingly, the Tonnetz of Figure 29 is quite similar to a structure discovered in the very early days of neo-Riemannian theory. Figure 34 presents an annotated reproduction of Ed Gollin's "3D Tonnetz," consisting of a series of planes each containing the notes of a diminished-seventh chord (Gollin 1998).[49] Exactly as in our own four-note Tonnetz, each note is connected to all the notes on the next plane except those a semitone away. The differences are relatively minor. First, Gollin's space contains a few superfluous connections. For example, the C in the diminished-seventh chord is connected by two distinct line segments to F♯/G♭, implying that there are two different but equal ways to move between them; in our space, by contrast, the diminished-seventh chord forms a tetrahedron, with exactly one line segment connecting any two notes.[50] Second, Gollin describes his figure as a

48 This graph can be obtained from the familiar chain of four-dimensional cubes (Figure 17) by eliminating diminished-seventh chords and connecting dominant and half-diminished sevenths when they share a diminished triad.

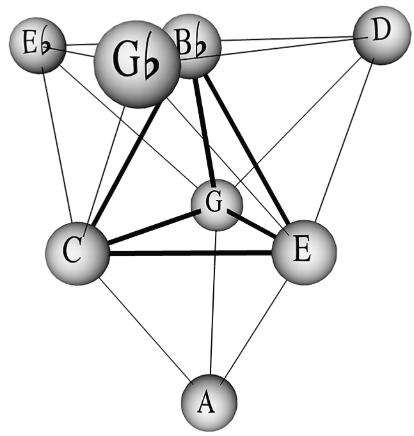
49 In an unpublished 1998 letter, Jack Douthett extended Gollin's figure by including minor-seventh chords, coming very close to the four-dimensional Tonnetz (which, unlike Douthett's graph, also has French sixths).

50 On a sphere, any two points can be connected by two separate arcs of the same great circle. But if Gollin's cross sections were truly spherical, then some of these line segments should intersect each other.

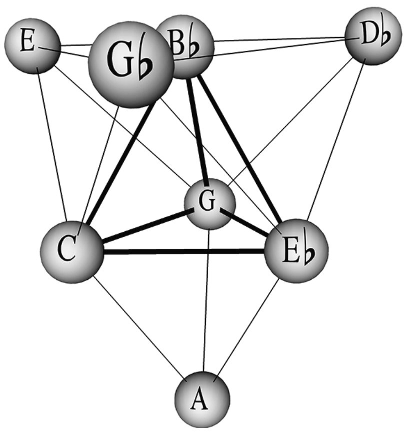
(a)



(b)



(c)



(d)

C^7	c^7	$C^{7[\flat 5]}$	$C^{ø7}$
C/D	C/D♭	C/D♭	C/D♭
E/E♭	E♭/E	E/E♭	E♭/E
G/G♭	G/G♭	G♭/G	G♭/G
B♭/A	B♭/A	B♭/A	B♭/A♭

Figure 31. (a–c) The local geometry around all chords on the Tonnetz is identical. (d) Every note of every chord can either be raised or lowered but not both.

"three-dimensional torus," whereas our figure is (topologically, at least) the twisted product of a circle and a two-dimensional sphere. Third, Gollin proposes his structure as a model of voice-leading relationships between dominant seventh chords and half-diminished sevenths only, whereas our figure also contains minor sevenths and French sixths. (Moreover, the note-based graph in Figure 16 includes diminished sevenths as well.) These differences notwithstanding, the relationship between the two figures is actually quite remarkable. Gollin may not have had a robust geometrical framework for thinking through these issues, but he came very close to the figure that we have just described.

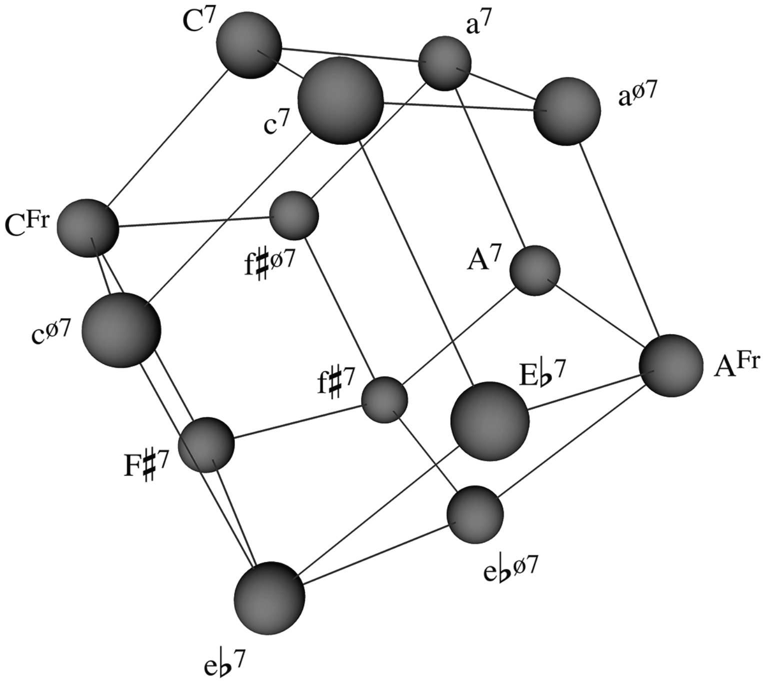


Figure 32. The dual of the four-dimensional Tonnetz is a circle of rhombic dodecahedra. Each dodecahedron contains four dominant sevenths, half-diminished sevenths, and minor sevenths, as well as two French sixths. Connections between dodecahedra occur by way of the tetrachordal "R relation"—single-step voice leading between dominant and half-diminished sevenths sharing a diminished triad.

As before, the ideas in this section can be extended to arbitrary dimensions. Whenever the number of notes in our chord evenly divides the number of notes in our scale, we can construct a chord-based lattice that is a circle of n-dimensional cubes linked by shared vertices. Taking the dual of each hypercube and attaching in the appropriate way, we produce a circle of n-dimensional cross-polytopes linked by shared simplicial facets. We then forget the perfectly even chord, reconceiving the graph's topology by "flattening" the shared $(n-1)$-simplicial facets so that they lie within an $(n-2)$-dimensional spherical space. This "forgetting" of the perfectly even chord has the effect of linking chords by a generalized version of the "R relation"—linking chords such as major and minor triads, or dominant and half-diminished sevenths, that share all but one of their notes with the perfectly even chord.[51] The resulting graph can be conceived as a circular arrangement

51 The original circle of cross-polytopes will require flip restrictions, since these chords share all but one of their notes *both* with each other and with the perfectly even chord. To model voice leading, we therefore need to require that they move to one another not directly but by way of the perfectly even chord. By forgetting the perfectly even chord, we lose the intermediary and hence the need for flip restrictions—at the cost of distorting voice-leading distances.

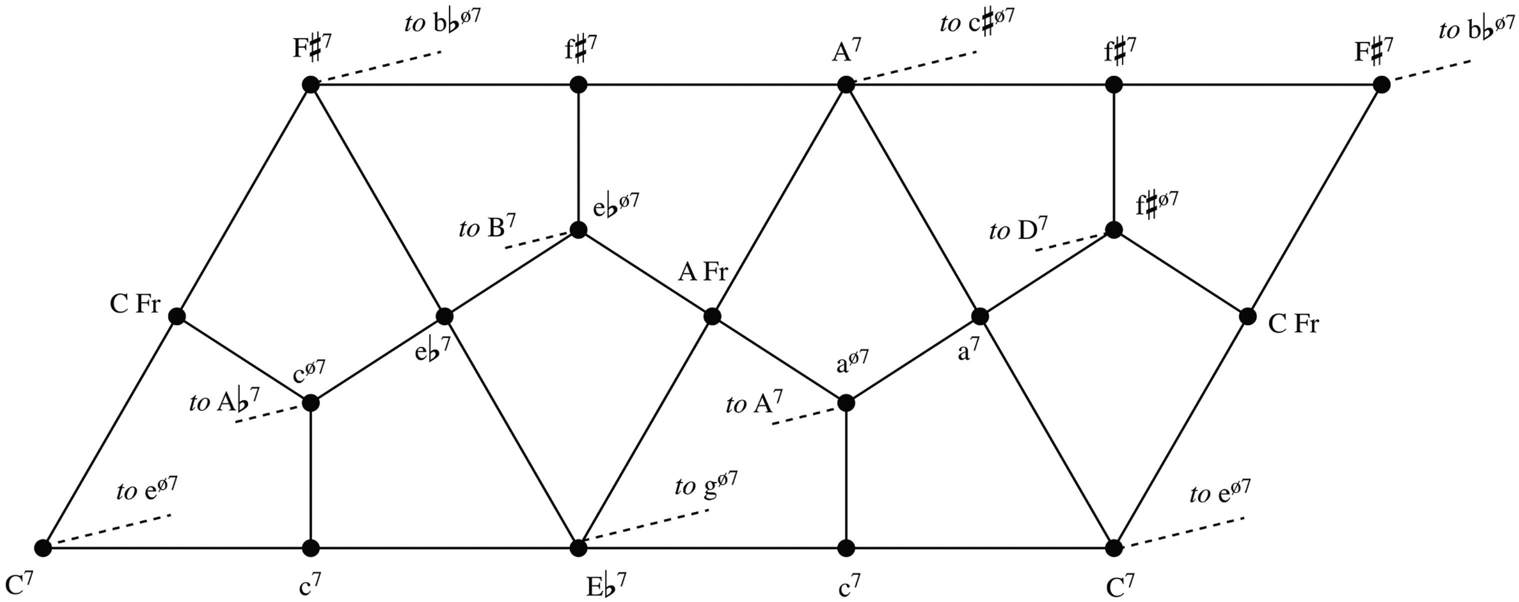


Figure 33. Another look at the cross section of the four-note Tonnetz's dual graph (a rhombic dodecahedron). Dominant and half-diminished sevenths are connected to chords in other cross sections, represented by dotted lines. If one follows the boundary from A^7 to E♭^{7}, one obtains the same chords no matter in which direction one proceeds. This implies that the graph can be embedded on a two-dimensional sphere (Figure 32).

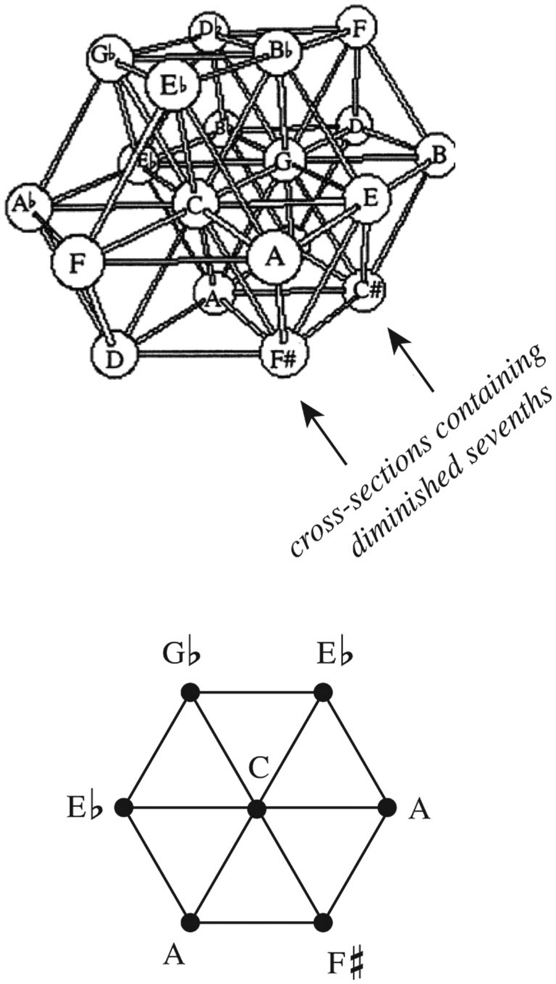


Figure 34. (Top) Ed Gollin's three-dimensional Tonnetz (1998) is remarkably similar to the one we are discussing (Figure 29). Gollin's graph consists of a series of diminished-seventh layers, with each note connected to all the other notes in its own layer, and to all the notes in the adjacent layers except those a semitone away. (Bottom) The only difference lies in the internal geometry of the layers, with Gollin's cross section not being tetrahedral. However, one can see hints that the cross section should be spherical, since the two circumferential paths from G♭ to F♯ pass through the same notes in the same order. (Gollin incorrectly describes this graph as toroidal.) It is likely that Gollin was trying to create *straight lines* that correspond to motion by particular intervals (e.g., perfect fifth, major third, etc.), but as in the case of the three-note Tonnetz, this is incompatible with the goal of representing voice leading accurately.

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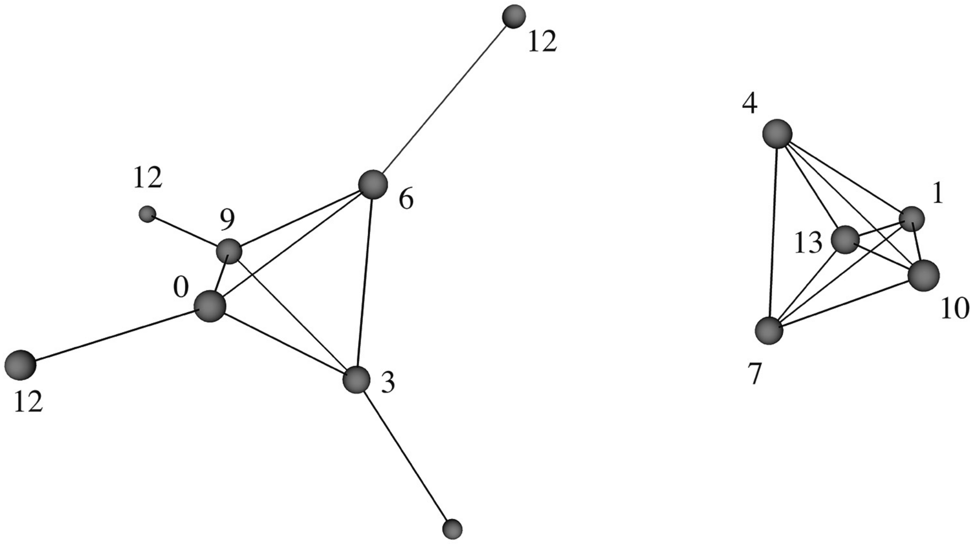

Figure 35. Two adjacent layers of the five-note Tonnetz. Each four-dimensional simplex is "flattened" into a three-dimensional figure, analogous to the two tetrahedra in Figure 28. Numbers refer to scale degrees in fifteen-tone equal temperament. Each note on one layer is connected to all the notes on the other layer, except for the one a semitone away.

of layers, with each layer being an $(n - 2)$-dimensional sphere, topologically the outside or "hull" of an $(n - 1)$-dimensional simplex, and with every note on one layer connected to all the notes on the next layer *except* for those that are a single scale step away. Thus, the generalized Tonnetz, rather than being a higher-dimensional torus, is the twisted product of a circle and a higher-dimensional sphere—mathematically, an $\mathbb{S}^{n-2}$ bundle over $\mathbb{S}^1$.[52] (The two-dimensional Tonnetz is a series of perfect fifths linked by common tones—Figure 13c without the vertical lines; the six-dimensional Tonnetz is graph-theoretically identical to Walter O'Connell's "tone lattice" [1968], the *complete graph* of pitch classes.) Figure 35 presents two adjacent layers of the Tonnetz representing five-note chords in the fifteen-note equal-tempered scale.

5. Historical and Analytical Conclusion

A decade ago, theorists confronted a blizzard of seemingly unrelated graphs. Besides the standard Tonnetz, there was Ed Gollin's three-dimensional Tonnetz (Figure 34); Douthett and Steinbach's (1998) chord-based "chicken-wire

52 The graph is "twisted" for the same reason that the chord-based analogue is twisted: a series of transpositions will return us to the same horizontal location but in a new spatial orientation. Thus, for instance, the top and bottom faces of Figure 14b are rotated relative to one another: if we transpose the augmented triad up by four steps, we return to the same triangle but with B in the voice that held G, D♯ in the voice that held B, and so on.

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torus" (Figure 1b), Cube Dance (Figure 14a), and "Power Towers" (a subgraph of Figure 17); John Roeder's set-class graphs; Cliff Callender's trichordal set-class space; Richard Cohn's tetrahedral set-class space; and Ian Quinn's six-dimensional, Fourier-based model of chord quality.[53] To the casual—or even committed—theorist, it was not clear where these models came from or how they related to one another. What was needed was a twofold process of generalization, one that allowed us to extend these specific models to a wider range of musical circumstances (including arbitrary chords in arbitrary scales), while also uncovering the structural principles linking them.

Since then we have seen significant progress on both fronts. An early step was describing the continuous spaces representing voice-leading relationships among all *n*-note chords, spaces that naturally contained chord-based graphs such as Cube Dance and Power Towers (Tymoczko 2006, 2011). From there, it was possible to understand the analogous "set-class" graphs discussed by Roeder, Callender, and Cohn (Callender, Quinn, and Tymoczko 2008). (Indeed, these set-class graphs are essentially projections of chord spaces along the direction representing transposition.) Clear understanding of these set-class graphs in turn made it possible to draw connections to Quinn's Fourier spaces.[54] With this article, we can start to bring the note-based graphs into the fold, for we now have the ability to produce Tonnetz-style graphs that describe a wide range of musical circumstances, as well as a more principled understanding of their relation to their chord-based cousins. In this sense, we are nearing the point where we can begin to see the outlines of a complete geometrical theory of voice-leading.

Particularly interesting here is the way the Tonnetz, a fundamentally *discrete* structure, falls out of the *continuous* spaces representing all possible three-note chords. When Callender, Quinn, and I were struggling to formulate our general approach to chord and set-class geometry, continuity was an important methodological principle: a robustly physical fact—since frequency is in fact continuous—that privileges certain music-theoretical constructions over others. (Indeed, continuity was a key feature of Callender's groundbreaking 2004 paper.) For instance, continuity leads us to consider the (note-based) circle of semitones B–C–C♯– . . . –[B] more fundamental than the circle of fifths B–F♯–C♯– . . . –[B], since the former, but not the latter, is a simple discretization of the continuous pitch-class circle. From this point of view, it is gratifying to find the circle of fifths reappearing as a *chord-based graph* describing single-semitone voice leading among diatonic scales.[55] But the Tonnetz never made any such reappearance: at best, it seemed like an

53 See Gollin 1998, Douthett and Steinbach 1998, Roeder 1984 and 1987, Callender 2004, Cohn 2003, and Quinn 2006 and 2007.

54 See Callender 2007, Hoffmann 2008, and Tymoczko 2008c.

55 The circle of fifths can also be understood as the two-note analogue of the traditional Tonnetz: if we remove the tritones from Figure 13c, we obtain a sequence of perfect fifths linked by voice leading in which only one voice moves, and it moves by a major second.

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inaccurate and incomplete version of the lattice at the center of chromatic three-note chord space rather than something more principled. Now, however, we understand that there is a nontoroidal version of the Tonnetz that is simply the geometrical dual of Cube Dance, containing augmented triads and faithfully representing voice-leading distances among its constituent chords. This allows us to derive the Tonnetz from the continuous geometrical spaces representing chords in general.

What remains is the very large project of using these spaces to elucidate particular pieces. From this perspective, the work in this article might seem somewhat superfluous, since it simply provides alternative representations of relationships already modeled by the well-understood family of chord-based graphs. However, there are situations where note-based graphs are quite useful. Cohn, for example, has stressed that these graphs can sometimes allow analysts to track the play of pitch classes more easily than the chord-based alternatives. This is relevant in music where common-tone relationships play an important role.[56] There is also the fact that the three- and four-note Tonnetze can be embedded in spaces of one fewer dimension than their chord-based counterparts: the standard Tonnetz is embeddable in two-dimensional toroidal space, in either its note-based or chord-based versions (Figure 1a,b), whereas "Cube Dance" requires three dimensions (Figure 14a); similarly, the four-note Tonnetz (Figure 27) is embeddable in a three-dimensional space, whereas the lattice at the center of four-note chord-space requires four dimensions. (This reduction in dimensionality, as we have seen, is a byproduct of the way the graphs eliminate the perfectly even chord, subtly distorting voice-leading distances.) Not only does this dimensional reduction aid in visualization, but it also opens the door to some interesting theoretical questions. Giovanni Albini, for example, has explored Hamiltonian paths through these spaces both in compositions and in theoretical work. (A *Hamiltonian path* touches on all the vertices in a graph without passing through any of them twice, and is in that sense a generalization of the twelve-tone row. If we are interested in Hamiltonian paths, it pays to remove the augmented triad and diminished-seventh chords, as they severely constrain the possibilities.) Finally, it is inherently useful to have a principled theoretical understanding of the connections between our various geometrical models of chord structure. The very existence of these multiple models testifies to the incredible richness of the voice-leading relations that underwrite so much familiar music.

That said, the complications we have encountered do underscore the simplicity of the chord-based models. It is no accident, I think, that a general understanding of voice-leading geometry began with the chord-based spaces, as they are in many ways simpler to construct, generalize, and comprehend.

56 Suzannah Clark has argued that such relationships are particularly important in Schubert, a point echoed in Cohn's own analysis of "Der Doppelgänger." See Clark 2002 and Cohn 2011b.

Absent a robust understanding of the full family of chord-based graphs, it would be quite hard to see that the nontoroidal, three-dimensional Tonnetz (Figure 14b) is in some ways preferable to the toroidal version shown in Figure 1a. Similarly, it would be (and indeed was) difficult to realize that the generalized Tonnetz is the twisted product of a circle and a higher-dimensional sphere. Furthermore, note-based graphs confront some inherent limitations: as we consider more and more chords of larger and larger size, the note-based Tonnetze become harder and harder to use, not just because their dimension increases but also because they represent chords using extended shapes (polytopes) which inevitably become hard to visualize. (The ratio of polytopes to vertices grows with the dimension of the space, requiring us to picture increasingly complex arrangements of the same basic pitch classes.)[57] Finally, while it is possible to eliminate dimensions in the second family of chord-based graphs, this is not true in the note-based case: the chord-based circle of fifths (Figure 19) is one-dimensional, while the note-based alternative requires *five additional dimensions*! And of course, there are also those unattractive but unavoidable redundancies and flip restrictions. For all of these reasons, chord-based graphs are significantly simpler and more straightforward than the Tonnetze we have explored in this article.

Rather than concluding with a decision in favor of one or another type of lattice, however, let me instead close by reflecting on the amazing fact that we can derive something like the standard Tonnetz in three very different ways: as a graph of acoustical relations among notes, as a graph of common-tone relations among triads, and as a graph of efficient voice leading among nearly even three-note chords. Cohn (2011a) has emphasized the many different times in which the Tonnetz has been rediscovered, by theorists with many different interests and agendas. The present article identifies yet another route to the figure: to generate Figure 14b's "chain of octahedra," we do not need to make any postulates about acoustics or common tones; instead, we simply take the geometrical dual of each cube in Figure 14a. In much the same way, we can derive the "common tone" and "acoustical" Tonnetze without mentioning voice leading at all. Prior to writing this article, I would have said that *one and the same structure* could be put to three different theoretical uses. And in a sense this is true: from the standpoint of graph theory, there is just one equal-tempered Tonnetz. But to the geometrical music theorist, the acoustic, common-tone, and voice-leading Tonnetze are all subtly different creatures. The coincidence among them was striking enough to mislead the theoretical community into thinking that there was just one underlying structure, but we can now see that this is wrong. Our quest to generalize the Tonnetz has also led us to particularize it, teasing apart the very similar structures that have previously gone under a single name.

57 For instance, the graph in Figure 14b has twelve points and thirty-six triangles, while that in Figure 16 has twelve points and forty-six tetrahedra.

Works Cited

Brower, Candace. 2008. “Paradoxes of Pitch Space.” *Music Analysis* 27/1: 51–106.

Callender, Clifton. 2004. “Continuous Transformations.” *Music Theory Online* 10/3. www.mtosmt.org/issues/mto.04.10.3/mto.04.10.3.callender.pdf.

———. 2007. “Continuous Harmonic Spaces.” *Journal of Music Theory* 41: 277–32.

Callender, Clifton, Ian Quinn, and Dmitri Tymoczko. 2008. “Generalized Voice Leading Spaces.” *Science* 320: 346–48.

Chew, Elaine. 2000. *Towards a Mathematical Model of Tonality*. Ph.D. diss., MIT.

Clark, Suzannah. 2002. “Schubert, Theory, and Analysis.” *Music Analysis* 21/2: 209–43.

Clough, John, and Jack Douthett. 1991. “Maximally Even Sets.” *Journal of Music Theory* 35: 93–173.

Clough, John, and Gerald Myerson. 1985. “Variety and Multiplicity in Diatonic Systems.” *Journal of Music Theory* 29/2: 249–70.

Cohn, Richard. 1996. “Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions.” *Music Analysis* 15/1: 9–40.

———. 1997. “Neo-Riemannian Operations, Parsimonious Trichords, and Their ‘Tonnetz’ Representations.” *Journal of Music Theory* 41/1: 1–66.

———. 2003. “A Tetrahedral Model of Tetrachordal Voice-Leading Space.” *Music Theory Online* 9/4. www.mtosmt.org/issues/mto.03.9.4/mto.03.9.4.cohn.frames.html.

———. 2011b. “Tonal Pitch Space and the (Neo-)Riemannian Tonnetz.” In *The Oxford Handbook of Neo-Riemannian Music Theories*, ed. Ed Gollin and Alexander Rehdning, 322–48. New York: Oxford University Press.

———. 2011a. *Audacious Euphony*. New York: Oxford University Press.

Cook, Robert. 2005. “Parsimony and Extravagance.” *Journal of Music Theory* 49/1: 109–40.

Douthett, Jack. 2008. “Filtered Point Symmetry and Dynamical Voice Leading.” In *Music Theory and Mathematics: Chords, Collections, and Transformations*, ed. Jack Douthett, Martha Hyde, and Charles Smith, 72–136. Rochester: University of Rochester Press.

Douthett, Jack, and Peter Steinbach. 1998. “Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition.” *Journal of Music Theory* 42/2: 241–63.

Gollin, Edward. 1998. “Some Aspects of Three-Dimensional ‘Tonnetze.’” *Journal of Music Theory* 42/2: 195–206.

Hall, Rachel, and Dmitri Tymoczko. 2012. “Submajorization and the Geometry of Unordered Collections.” *American Mathematical Monthly*. 119/4: 263–83.

Hoffman, Justin. 2008. “On Pitch-Class Set Cartography: Relations between Voice-Leading Spaces and Fourier Spaces.” *Journal of Music Theory* 52: 219–49.

Jackson, Allyn. 2002. “The World of Blind Mathematicians.” *Notices of the American Mathematical Society* 49/10: 1246–51.

Lewin, David. 1987. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press.

O’Connell, Walter. 1968 [1962]. “Tone Spaces.” Rev. English ed. *Die Reihe* 8: 3467.

Quinn, Ian. 2006. “General Equal-Tempered Harmony.” *Perspectives of New Music* 44/2: 5–60.

———. 2007. “General Equal-Tempered Harmony: Part II.” *Perspectives of New Music* 45/1: 6–65.

Roeder, John. 1984. “A Theory of Voice Leading for Atonal Music.” Ph.D. diss., Yale University.

———. 1987. “A Geometric Representation of Pitch-Class Series.” *Perspectives of New Music* 25/1–2: 362–409.

Tymoczko, Dmitri. 2006. “The Geometry of Musical Chords.” *Science* 313: 72–74.

———. 2008a. “Lewin, Intervals, and Transformations: A Response to Hook.” *Music Theory Spectrum* 30/1: 164–68.

———. 2008b. "Scale Theory, Serial Theory, and Voice Leading." *Music Analysis* 27/1: 1–49.
———. 2008c. "Set-Class Similarity, Voice Leading, and the Fourier Transform." *Journal of Music Theory* 52/2: 251–72.
———. 2009b. "Three Conceptions of Musical Distance." In *Mathematics and Computation in Music*, ed. Elaine Chew, Adrian Childs, and Ching-Hua Chuan, 258–73. Heidelberg: Springer.
———. 2009a. "Generalizing Musical Intervals." *Journal of Music Theory* 53/2: 227–54.
———. 2010. "Geometrical Methods in Recent Music Theory." *Music Theory Online* 16/1. www.mtosmt.org/issues/mto.10.16.1/mto.10.16.1.tymoczko.html.
———. 2011. *A Geometry of Music*. New York: Oxford University Press.
———. Forthcoming. "Hey Wait a Minute." *Music Theory Spectrum*.

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