Ellipsoidally Symmetric Distributions

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Outline

The Ellipsoidally Symmetric Distribution

Definition

Let $R \subseteq [0,\infty)$, $f : [0,\infty) \to [0,\infty)$, and A an $N \times N$ positive definite symmetric matrix. A density function $g : R^N \to [0,\infty)$ is an ellipsoidally symmetric distribution if and only if

$$g(x) \quad = \quad cf(\sqrt{x'Ax})$$

where

$$\int_{y \in R} y^n f(y) dy < \infty, \; n \leq N+1$$

$$c \quad = \quad \frac{1}{\int_{\sqrt{x'Ax} \in R} f(\sqrt{x'Ax}) dx}$$

Determining Value of the Constant

$$\int_{\sqrt{x'Ax} \in R} f(\sqrt{x'Ax} dx$$

Let T be an orthonormal matrix satisfying

$$T'AT = D$$

where D is a diagonal matrix with no non-positive diagonal entries. Make the change of variables

$$x = TD^{-\frac{1}{2}} z$$

The Diagonalizing Transformation

$$x = TD^{-\frac{1}{2}}z$$

where

$$T'AT = D$$

Then,

$$\begin{aligned} x'Ax &= (TD^{-\frac{1}{2}}z)'A(TD^{-\frac{1}{2}})z \\ &= z'D^{-\frac{1}{2}}(T'AT)D^{-\frac{1}{2}}z \\ &= z'D^{-\frac{1}{2}}DD^{-\frac{1}{2}}z \\ &= z'z \end{aligned}$$

The Jacobian

$$x = TD^{-\frac{1}{2}}z$$

The determinant of the Jacobian of this transformation is $|A|^{-\frac{1}{2}}$ which is positive definite since A is positive definite.

$$\begin{aligned} J &= \frac{\partial x}{\partial z} \\ &= TD^{-\frac{1}{2}} \end{aligned}$$

Jacobian Determinant

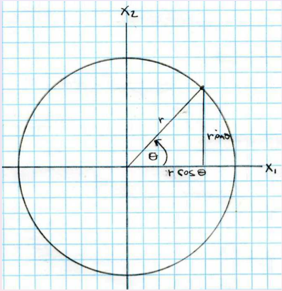
The determinant of the Jacobian of this transformation is $|A|^{-\frac{1}{2}}$ which is positive definite since A is positive definite.

$$
\begin{aligned}
|J| &= |TD^{-\frac{1}{2}}| \\
&= |T|\,|D^{-\frac{1}{2}}| \\
&= |D^{-\frac{1}{2}}| \\
&= |D|^{-\frac{1}{2}} \\
&= |T'AT|^{-\frac{1}{2}} \\
&= (|T'|\,|A|\,|T|)^{-\frac{1}{2}} \\
&= |A|^{-\frac{1}{2}}
\end{aligned}
$$

The Diagonalizing Transformation

$$\int_{\sqrt{x'Ax} \in R} f(\sqrt{x'Ax})\, dx \quad = \quad |A|^{-\frac{1}{2}} \int_{\sqrt{z'z} \in R} f(\sqrt{z'z}) dz$$

Polar Coordinate System



$$\begin{aligned} z_1 &= r\cos\theta_1 \\ z_2 &= r\sin\theta_1 \end{aligned}$$

Area

Let A be the area of a circle of radius r_0.

$$A = \int_{\sqrt{z_1^2+z_2^2} \leq r_0} dz_1 dz_2$$

$$\begin{aligned}
z_1 &= r\cos\theta_1 \\
z_2 &= r\sin\theta_1 \\
z_1^2 + z_2^2 &= (r\cos\theta_1)^2 + (r\sin\theta_1)^2 \\
&= r^2(\cos^2\theta_1 + \sin^2\theta_1) \\
&= r^2
\end{aligned}$$

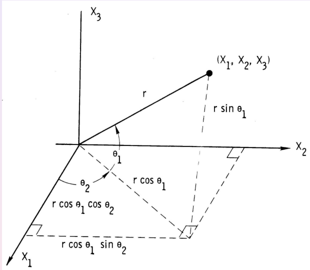
Transformation to Polar Coordinates

$$
\begin{aligned}
z(r,\theta_1) &= \begin{pmatrix} z_1(r,\theta_1) \\ z_2(r,\theta_1) \end{pmatrix} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} \\
J_z(r,\theta_1) &= \begin{pmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta_1} \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta_1} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{pmatrix} \\
|J_z(r,\theta_1)| &= r\cos\theta_1\cos\theta_1 + r\sin\theta_1\sin\theta_1 \\
&= r
\end{aligned}
$$

Area

$$
\begin{aligned}
A &= \int_{\sqrt{z_1^2+z_2^2}\leq r_0} dz_1 dz_2 \\
&= \int_{r=0}^{r_0}\int_{\theta_1=0}^{2\pi} |J_z(r,\theta_1)| dr d\theta_1 \\
&= \int_{r=0}^{r_0}\int_{\theta_1=0}^{2\pi} r dr d\theta_1 \\
&= \int_{r=0}^{r_0} r dr \int_{\theta_1=0}^{2\pi} d\theta_1 \\
&= \left[\frac{r^2}{2}\right]_0^{r_0} [\theta_1]_0^{2\pi} = \frac{r_0^2}{2} 2\pi = \pi r_0^2
\end{aligned}
$$

Three Dimensional Spherical Coordinates



$$
\begin{aligned}
z_1 &= r\cos\theta_1\cos\theta_2 \\
z_2 &= r\cos\theta_1\sin\theta_2 \\
z_3 &= r\sin\theta_1
\end{aligned}
$$

Volume Integral

Let V be the volume of a sphere of radius r_0.

$$V = \int_{\sqrt{z'z} \leq r_0} dz$$

$$\begin{aligned} z_1 &= r \cos\theta_1 \cos\theta_2 \\ z_2 &= r \cos\theta_1 \sin\theta_2 \\ z_3 &= r \sin\theta_1 \end{aligned}$$

$$z'z = z_1^2 + z_2^2 + z_3^2$$

The Quadratic Form

$$
\begin{aligned}
z_1 &= r\cos\theta_1\cos\theta_2 \\
z_2 &= r\cos\theta_1\sin\theta_2 \\
z_3 &= r\sin\theta_1
\end{aligned}
$$

$$
\begin{aligned}
z_1^2 + z_2^2 + z_3^2 &= (r\cos\theta_1\cos\theta_2)^2 + (r\cos\theta_1\sin\theta_2)^2 + (r\sin\theta_1)^2 \\
&= r^2(\cos^2\theta_1\cos^2\theta_2 + \cos^2\theta_1\sin^2\theta_2 + \sin^2\theta_1 \\
&= r^2\left(\cos^2\theta_1(\cos^2\theta_2 + \sin^2\theta_2) + \sin^2\theta_1\right) \\
&= r^2(\cos^2\theta_1 + \sin^2\theta_1) \\
&= r^2
\end{aligned}
$$

The Transformation to Spherical Cooridinates

$$z(r,\theta_1,\theta_2) = \begin{pmatrix} z_1(r,\theta_1,\theta_2) \\ z_2(r,\theta_1,\theta_2) \\ z_3(r,\theta_1,\theta_2) \end{pmatrix} = \begin{pmatrix} r\cos\theta_1\cos\theta_2 \\ r\cos\theta_1\sin\theta_2 \\ r\sin\theta_1 \end{pmatrix}$$

The Jacobian

$$z(r,\theta_1,\theta_2) = \begin{pmatrix} z_1(r,\theta_1,\theta_2) \\ z_2(r,\theta_1,\theta_2) \\ z_3(r,\theta_1,\theta_2) \end{pmatrix} = \begin{pmatrix} r\cos\theta_1\cos\theta_2 \\ r\cos\theta_1\sin\theta_2 \\ r\sin\theta_1 \end{pmatrix}$$

The Jacobian, denoted by J_z is defined by

$$\begin{aligned} J_z(r,\theta_1,\theta_2) &= \frac{\partial(z_1,z_2,z_3)}{\partial(r,\theta_1,\theta_2)} \\ &= \begin{pmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta_1} & \frac{\partial z_1}{\partial \theta_2} \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta_1} & \frac{\partial z_2}{\partial \theta_2} \\ \frac{\partial z_3}{\partial r} & \frac{\partial z_3}{\partial \theta_1} & \frac{\partial z_3}{\partial \theta_2} \end{pmatrix} \end{aligned}$$

The Jacobian

$$J_z = \begin{pmatrix} \cos\theta_1\cos\theta_2 & -r\sin\theta_1\cos\theta_2 & -r\cos\theta_1\sin\theta_2 \\ \cos\theta_1\sin\theta_2 & -r\sin\theta_1\sin\theta_2 & r\cos\theta_1\cos\theta_2 \\ \sin\theta_1 & r\cos\theta_1 & 0 \end{pmatrix}$$

The determinant $|J_z|$ of the Jacobian J_z is $-r^2\cos\theta_1$

The Volume Integral

$$
\begin{aligned}
V &= \int_{\sqrt{z'z} \le r_0} dz \\
&= \int_{r \le r_0} |J_z| dr \; d\theta_1 \; d\theta_2 \\
&= \int_{r=0}^{r_0} \int_{\theta_1=-\pi/2}^{\pi/2} \int_{\theta_2=0}^{2\pi} |-r^2 \cos\theta_1| dr \; d\theta_1 \; d\theta_2 \\
&= \int_{r=0}^{r_0} r^2 dr \int_{\theta_1=-\pi/2}^{\pi/2} \cos\theta_1 d\theta_1 \int_{\theta_2=0}^{2\pi} d\theta_2 \\
&= \left[\frac{r^3}{3}\right]_0^{r_0} [\sin\theta_1]_{-\pi/2}^{\pi/2} 2\pi = \frac{r_0^3}{3} \times 2 \times 2\pi = 4/3\, \pi r_0^3
\end{aligned}
$$

The Jacobian Determinant

$$J_z = \begin{pmatrix} \cos\theta_1 \cos\theta_2 & -r\sin\theta_1 \cos\theta_2 & -r\cos\theta_1 \sin\theta_2 \\ \cos\theta_1 \sin\theta_2 & -r\sin\theta_1 \sin\theta_2 & r\cos\theta_1 \cos\theta_2 \\ \sin\theta_1 & r\cos\theta_1 & 0 \end{pmatrix}$$

Function Composition

$$
\begin{aligned}
w(\rho, u, \phi_2) &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \rho\cos\phi_2 \\ \rho\sin\phi_2 \\ u \end{pmatrix} \\
v(r, \theta_1, \theta_2) &= \begin{pmatrix} \rho \\ u \\ \phi_2 \end{pmatrix} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \\ \theta_2 \end{pmatrix} \\
w(v(r, \theta_1, \theta_2)) &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} r\cos\theta_1\cos\theta_2 \\ r\cos\theta_1\sin\theta_2 \\ r\sin\theta_1 \end{pmatrix}
\end{aligned}
$$

The Jacobian J_w

$$
\begin{aligned}
w(\rho, u, \phi_2) &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \rho \cos \phi_2 \\ \rho \sin \phi_2 \\ u \end{pmatrix} \\
J_w &= \begin{pmatrix} \frac{\partial z_1}{\partial \rho} & \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial \phi_2} \\ \frac{\partial z_2}{\partial \rho} & \frac{\partial z_2}{\partial u} & \frac{\partial z_2}{\partial \phi_2} \\ \frac{\partial z_3}{\partial \rho} & \frac{\partial z_3}{\partial u} & \frac{\partial z_3}{\partial \phi_2} \end{pmatrix} \\
&= \begin{pmatrix} \cos \phi_2 & 0 & -\rho \sin \phi_2 \\ \sin \phi_2 & 0 & \rho \cos \phi_2 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

Determinant of J_w

$$\begin{aligned}
J_w &= \begin{pmatrix} \cos\phi_2 & 0 & -\rho\sin\phi_2 \\ \sin\phi_2 & 0 & \rho\cos\phi_2 \\ 0 & 1 & 0 \end{pmatrix} \\
|J_w| &= -\begin{vmatrix} \cos\phi_2 & -\rho\sin\phi_2 \\ \sin\phi_2 & \rho\cos\phi_2 \end{vmatrix} \\
&= -(\rho\cos\phi_2\cos\phi_2 + \rho\sin\phi_2\sin\phi_2) \\
&= -\rho(\cos^2\phi_2 + \sin^2\phi_2) \\
&= -\rho
\end{aligned}$$

The Jacobian J_v

$$v(r, \theta_1, \theta_2) = \begin{pmatrix} \rho \\ u \\ \phi_2 \end{pmatrix} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \\ \theta_2 \end{pmatrix}$$

$$J_v = \begin{pmatrix} \frac{\partial \rho}{\partial r} & \frac{\partial \rho}{\partial \theta_1} & \frac{\partial \rho}{\partial \theta_2} \\ \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta_1} & \frac{\partial u}{\partial \theta_2} \\ \frac{\partial \phi_2}{\partial \theta_2} & \frac{\partial \phi_2}{\partial \theta_2} & \frac{\partial \phi_2}{\partial \theta_2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1 & r\cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Determinant of J_v

$$
\begin{aligned}
J_v &= \begin{pmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1 & r\cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
|J_v| &= \begin{vmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{vmatrix} \\
&= r\cos\theta_1\cos\theta_1 + r\sin\theta_1\theta_1 \\
&= r(\cos^2\theta_1 + \sin^2\theta_1) \\
&= r
\end{aligned}
$$

Chain Rule

$$\begin{aligned}
J_{wv} &= J_w J_v \\
&= \begin{pmatrix} \cos\phi_2 & 0 & -\rho\sin\phi_2 \\ \sin\phi_2 & 0 & \rho\cos\phi_2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1 & r\cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta_1\cos\phi_2 & -r\sin\theta_1\cos\phi_2 & -\rho\sin\theta_2 \\ \cos\theta_1\sin\phi_2 & -r\sin\theta_1\sin\phi_2 & \rho\cos\theta_2 \\ \sin\theta_1 & r\cos\theta_1 & 0 \end{pmatrix}
\end{aligned}$$

But $\rho = r\cos\theta_1$ and $\phi_2 = \theta_2$.

$$J_{wv} = \begin{pmatrix} \cos\theta_1\cos\theta_2 & -r\sin\theta_1\cos\theta_2 & -r\cos\theta_1\sin\theta_2 \\ \cos\theta_1\sin\theta_2 & -r\sin\theta_1\sin\theta_2 & r\cos\theta_1\cos\theta_2 \\ \sin\theta_1 & r\cos\theta_1 & 0 \end{pmatrix}$$

Jacobian Determinant

$$\begin{aligned} J_z &= J_{wv} = J_w J_v \\ |J_z| &= |J_w|\,|J_v| \\ &= -\rho r \end{aligned}$$

But $\rho = r\cos\theta_1$

$$\begin{aligned} |J_z| &= -\rho r \\ &= -r\cos\theta_1 r \\ &= -r^2\cos\theta_1 \end{aligned}$$

Changing To N-Dimensional Spherical Coordinates

$$
\begin{aligned}
z_1 &= r\cos\theta_1\cos\theta_2 \ldots\ldots\ldots\ldots\ldots \cos\theta_{N-2}\cos\theta_{N-1} \\
z_2 &= r\cos\theta_1\cos\theta_2 \ldots\ldots\ldots\ldots\ldots \cos\theta_{N-2}\sin\theta_{N-1} \\
z_3 &= r\cos\theta_1\cos\theta_2 \ldots \cos\theta_{N-3}\sin\theta_{N-2} \\
&\vdots \\
z_{N-2} &= r\cos\theta_1\cos\theta_2\sin\theta_3 \\
z_{N-1} &= r\cos\theta_1\sin\theta_2 \\
z_N &= r\sin\theta_1
\end{aligned}
$$

Changing to N-Dimensional Spherical Coordinates

$$z(r, \theta_1 \ldots, \theta_{N-1}) = \begin{pmatrix} r\cos\theta_1\cos\theta_2 \ldots\ldots\ldots\ldots \cos\theta_{N-2}\cos\theta_{N-1} \\ r\cos\theta_1\cos\theta_2 \ldots\ldots\ldots\ldots \cos\theta_{N-2}\sin\theta_{N-1} \\ r\cos\theta_1\cos\theta_2 \ldots \cos\theta_{N-3}\sin\theta_{N-2} \\ \vdots \\ r\cos\theta_1\cos\theta_2\sin\theta_3 \\ r\cos\theta_1\sin\theta_2 \\ r\sin\theta_1 \end{pmatrix}$$

The Jacobian of z is

$$J_z(r, \theta_1 \ldots, \theta_{N-1}) = J_N$$

Changing To N-Dimensional Spherical Coordinates

$$
\begin{aligned}
z_1 &= (r\cos\theta_1)\cos\phi_2 \ldots\ldots\ldots\ldots \cos\phi_{N-2}\cos\phi_{N-1} \\
z_2 &= (r\cos\theta_1)\cos\phi_2 \ldots\ldots\ldots\ldots \cos\phi_{N-2}\sin\phi_{N-1} \\
z_3 &= (r\cos\theta_1)\cos\phi_2 \ldots \cos\phi_{N-3}\sin\phi_{N-2} \\
&\vdots \\
z_{N-1} &= (r\cos\theta_1)\sin\phi_2 \\
z_N &= (r\sin\phi_1)
\end{aligned}
$$

Let

$$
\begin{aligned}
\rho &= r\cos\theta_1 \\
u &= r\sin\theta_1
\end{aligned}
$$

Changing To N-Dimensional Spherical Coordinates

$$
\begin{aligned}
w(\rho, u, \phi_2, \ldots, \phi_{N-1}) &= \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N-1} \\ z_N \end{pmatrix} \\
&= \begin{pmatrix} \rho \cos\phi_2 \ldots\ldots\ldots\ldots\ldots \cos\phi_{N-2} \cos\phi_{N-1} \\ \rho \cos\phi_2 \ldots\ldots\ldots\ldots\ldots \cos\phi_{N-2} \sin\phi_{N-1} \\ \rho \cos\phi_2 \ldots \cos\phi_{N-3} \sin\phi_{N-2} \\ \vdots \\ \rho \sin\phi_2 \\ u \end{pmatrix}
\end{aligned}
$$

The Jacobian of w

$$
\begin{aligned}
w(\rho, u, \phi_2, \ldots, \phi_{N-1}) &= \begin{pmatrix} \rho \cos\phi_2 \ldots\ldots\ldots\ldots \cos\phi_{N-2} \cos\phi_{N-1} \\ \rho \cos\phi_2 \ldots\ldots\ldots\ldots \cos\phi_{N-2} \sin\phi_{N-1} \\ \rho \cos\phi_2 \ldots \cos\phi_{N-3} \sin\phi_{N-2} \\ \vdots \\ \rho \sin\phi_2 \\ u \end{pmatrix} \\
|J_w| &= -|J_{N-1}(r\cos\theta_1, \phi_2 \ldots \phi_{N-1})| \\
&= -\cos^{N-2}\theta_1 |J_{N-1}(r, \phi_2 \ldots \phi_{N-1})|
\end{aligned}
$$

The Transformation v

$$
\begin{aligned}
v(r,\theta_1,\ldots\theta_{N-1}) &= \begin{pmatrix} \rho \\ u \\ \phi_2 \\ \vdots \\ \phi_{N-1} \end{pmatrix} \\
&= \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-1} \end{pmatrix}
\end{aligned}
$$

The Jacobian of v

$$
\begin{aligned}
v(r, \theta_1, \ldots \theta_{N-1}) &= \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-1} \end{pmatrix} \\
|J_v(r, \theta_1 \ldots \theta_{N-1})| &= \begin{vmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{vmatrix} \\
&= r
\end{aligned}
$$

The Determinant of the Jacobian of J_N

$$\begin{aligned}
J_N &= J_w J_v \\
|J_N| &= |J_w|\,|J_v| \\
&= r(-cos^{N-2}\theta_1 |J_{N-1}|) \\
&= r(-cos^{N-2}\theta_1 (r(-cos^{N-3}\theta_2 |J_{N-2}|) \\
&\vdots \\
&= (-1)^{N-1} r^{N-1} \prod_{n=1}^{N-2} \cos^{N-1-n} \theta_n
\end{aligned}$$

The Volume of a Hypersphere

$$V = \int_{\sqrt{z'z} \leq r_0} dz$$

Make a change to spherical coordinates

$$V = \int_{r=0}^{r_0} \int_{\theta_1=-\pi/2}^{\pi/2} \dots \int_{\theta_{N-2}=-\pi/2}^{\pi/2} \int_{\theta_{N-1}=0}^{2\pi} r^{N-1} \prod_{n=1}^{N-2} \cos^{N-1-n} \theta_n dr d\theta_1 \dots d\theta_{N-1}$$

$$= \int_{r=0}^{r_0} r^{N-1} dr \prod_{n=1}^{N-2} \int_{\theta_n=-\pi/2}^{\pi/2} cos^{N-1-n} \theta_n d\theta_n \int_{\theta_{N-1}=0}^{2\pi} d\theta_{N-1}$$

The Cosine Integrals

$$\int_{-\pi/2}^{\pi/2} \cos^k \theta d\theta \quad = \quad 2\int_0^{\pi/2} cos^k \theta d\theta$$

Let $u = \cos\theta$. Then $du = -sin\theta d\theta$.

$$\begin{aligned} d\theta &= \frac{-du}{\sin\theta} \\ &= -\frac{du}{\sqrt{1-\cos^2\theta}} \\ &= -\frac{du}{\sqrt{1-u^2}} \end{aligned}$$

The Cosine Integrals

$$\begin{aligned}
2\int_0^{\pi/2} cos^k\theta d\theta &= 2\int_1^0 u^k\left(-\frac{du}{\sqrt{1-u^2}}\right) \\
&= 2\int_0^1 u^k(1-u^2)^{-\frac{1}{2}}du \\
&= \int_0^1 u^{k-1}(1-u^2)^{-\frac{1}{2}}(2udu)
\end{aligned}$$

The Cosine Integrals

$$\int_{-\pi/2}^{\pi/2} \cos^k \theta d\theta = \int_0^1 u^{k-1}(1-u^2)^{-\frac{1}{2}}(2udu)$$

Let $v = u^2$. Then $dv = 2udu$

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos^k \theta d\theta &= \int_0^1 v^{\frac{k-1}{2}}(1-v)^{-\frac{1}{2}} dv \\ &= \int_0^1 v^{\frac{k+1}{2}-1}(1-v)^{\frac{1}{2}-1} dv \\ &= B(\frac{k+1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+2}{2})}\end{aligned}$$

The Cosine Integrals

$$\begin{aligned}
\prod_{n=1}^{N-2} \int_{\theta_n=-\pi/2}^{\pi/2} cos^{N-1-n}\theta_n d\theta_n &= \prod_{n=1}^{N-2} \frac{\Gamma(\frac{N-1-n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{N-1-n+2}{2})} \\
&= \Gamma(\frac{1}{2})^{N-2} \prod_{n=1}^{N-2} \frac{\Gamma(\frac{N-n}{2})}{\Gamma(\frac{N+1-n}{2})} \\
&= \Gamma^{N-2}(\frac{1}{2}) \frac{\Gamma(1)}{\Gamma(\frac{N}{2})} \\
&= \pi^{\frac{N-2}{2}} \frac{1}{\Gamma(\frac{N}{2})}
\end{aligned}$$

The Volume of a N-Dimensional Sphere

$$
\begin{aligned}
V &= \int_{r=0}^{r_0} r^{N-1} dr \prod_{n=1}^{N-2} \int_{\theta_n=-\pi/2}^{\pi/2} cos^{N-1-n}\theta_n d\theta_n \int_{\theta_{N-1}=0}^{2\pi} d\theta_{N-1} \\
&= \left[\frac{r^N}{N}\right]_0^{r_0} (\pi)^{\frac{N-2}{2}} \frac{1}{\Gamma(\frac{N}{2})} 2\pi \\
&= 2\frac{r_0^N \pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}
\end{aligned}
$$

Ellipsoidally Symmetric Distributions

$$\int_{\sqrt{x'Ax}\in R} f(\sqrt{x'Ax}dx = |A|^{-\frac{1}{2}} \int_{\sqrt{z'z}\in R} f(z'z)dz$$

Make a change to spherical coordinates.

$$\begin{aligned} |A|^{-\frac{1}{2}} \int_{\sqrt{z'z}\in R} f(\sqrt{z'z})dz &= 2\pi|A|^{-\frac{1}{2}} \int_{r\in R} f(r)r^{N-1}dr \\ & \prod_{n=1}^{N-2} \int_{-\pi/2}^{\pi/2} \cos^{N-1-n}\theta_n d\theta_n \end{aligned}$$

Ellipsoidally Symmetric Distributions

$$\begin{aligned}
|A|^{-\frac{1}{2}} \int_{\sqrt{z'z} \in R} f(\sqrt{z'z}) dz &= 2\pi |A|^{-\frac{1}{2}} \int_{r \in R} f(r) r^{N-1} dr \\
&\quad \prod_{n=1}^{N-2} \int_{-\pi/2}^{\pi/2} \cos^{N-1-n} \theta_n d\theta_n \\
&= 2\pi |A|^{-\frac{1}{2}} \int_{r \in R} f(r) r^{N-1} dr (\pi)^{\frac{N-2}{2}} \frac{1}{\Gamma(\frac{N}{2})} \\
&= \frac{2(\pi)^{\frac{N}{2}}}{|A|^{\frac{1}{2}} \Gamma(\frac{N}{2})} \int_{r \in R} r^{N-1} f(r) dr
\end{aligned}$$

Ellipsoidally Symmetric Distributions

The density function is

$$cf(\sqrt{x'Ax})$$

where

$$c = \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} \int_{r \in R} r^{N-1} f(r) dr}$$

Covariance

$$\Sigma = E[xx'] = c \int_{x'Ax \in R} xx' f(\sqrt{x'Ax} dx$$

Let T satisfy $T'AT = D$ where D is diagonal and T is orthonormal. Make the change of variables $x = TD^{-\frac{1}{2}} z$. Then

$$\Sigma = c|A|^{-\frac{1}{2}} TD^{-\frac{1}{2}} \int_{\sqrt{z'z} \in R} zz' f(\sqrt{z'z}) dz \; D^{-\frac{1}{2}} T'$$

Covariance

$$\Sigma = |A|^{-\frac{1}{2}} T D^{-\frac{1}{2}} c \int_{\sqrt{z'z} \in R} zz' f(\sqrt{z'z}) dz \; D^{-\frac{1}{2}} T'$$

Let

$$G = (g_{ij}) = c \int_{\sqrt{z'z} \in R} zz' f(\sqrt{z'z}) dz$$

Then for $i \neq j$

$$\begin{aligned} g_{ij} &= c \int_{\sqrt{z'z} \in R} z_i z_j f(\sqrt{z'z}) dz \\ &= 0 \end{aligned}$$

Covariance

$$
\begin{aligned}
g_{ii} &= c \int_{\sqrt{z'z} \in R} z_i^2 f(\sqrt{z'z}) dz \\
&= c \int_{\sqrt{z'z} \in R} z_1^2 f(\sqrt{z'z}) dz
\end{aligned}
$$

Make a change to the spherical coordinate system. Recall $z_1 = r \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{N-1}$.

Covariance

$$
\begin{aligned}
g_{ii} &= c\int_{r\in R}\int_{\theta_1=-\pi/2}^{\pi/2}\dots\int_{\theta_{N-2}=-\pi/2}^{\pi/2}\int_{\theta_{N-1}=0}^{2\pi}(r\prod_{n=1}^{N-1}\cos\theta_n)^2 f(r) \\
&\quad r^{N-1}\prod_{n=1}^{N-2}\int_{\theta_n=-\pi/2}^{\pi/2}\cos^{N-1-n}\theta_n d\theta_n dr d\theta_{N-1}\int_0^{2\pi}\cos^2\theta_{N-1}d\theta_{N-1} \\
&= \prod_{n=1}^{N-2}\int_{\theta_n=-\pi/2}^{\pi/2}\cos^{N+1-n}\theta_n d\theta_n c\int_{r\in R}r^{N+1}f(r)dr \\
&\quad 2\int_{-\pi/2}^{\pi/2}\cos^2\theta_{N-1}d\theta_{N-1}
\end{aligned}
$$

Covariance

$$\begin{aligned}
g_{ii} &= \prod_{n=1}^{N-2} \frac{\Gamma(\frac{N+2-n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{N+3-n}{2})} c \int_{r\in R} r^{N+1} f(r) dr \; 2 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{4}{2})} \\
&= \Gamma^{N-2}(\frac{1}{2}) \frac{\Gamma(\frac{4}{2})}{\Gamma(\frac{N+2}{2})} c \int_{r\in R} r^{N+1} f(r) dr \; 2 \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} \\
&= \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2})} c \int_{r\in R} r^{N+1} f(r) dr
\end{aligned}$$

Covariance

$$g_{ii} = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2})} c \int_{r \in R} r^{N+1} f(r) dr$$

Recall $c = \frac{|A|^{\frac{1}{2}} \Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} \int_{r \in R} r^{N-1} f(r) dr}$ Therefore,

$$g_{ii} = \frac{|A|^{\frac{1}{2}} \Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2})} \frac{\int_{r \in R} r^{N+1} f(r) dr}{\int_{r \in R} r^{N-1} f(r) dr}$$

Covariance

$$
\begin{aligned}
g_{ii} &= \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2})} \frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr} \\
&= \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\Gamma(\frac{N+2}{2})} \frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr} \\
&= \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\frac{N}{2}\Gamma(\frac{N}{2})} \frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr} \\
&= \frac{|A|^{\frac{1}{2}}}{N} \frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr}
\end{aligned}
$$

Covariance

$$
\begin{aligned}
\Sigma &= |A|^{-\frac{1}{2}} TD^{-\frac{1}{2}} c \int_{\sqrt{z'z} \in R} zz' f(\sqrt{z'z}) dz \, D^{-\frac{1}{2}} T' \\
&= |A|^{-\frac{1}{2}} TD^{-\frac{1}{2}} \frac{|A|^{\frac{1}{2}}}{N} \frac{\int_{r \in R} r^{N+1} f(r) dr}{\int_{r \in R} r^{N-1} f(r) dr} ID^{-\frac{1}{2}} T' \\
&= \frac{1}{N} \frac{\int_{r \in R} r^{N+1} f(r) dr}{\int_{r \in R} r^{N-1} f(r) dr} TD^{-\frac{1}{2}} D^{-\frac{1}{2}} T' \\
&= \frac{1}{N} \frac{\int_{r \in R} r^{N+1} f(r) dr}{\int_{r \in R} r^{N-1} f(r) dr} TD^{-1} T \\
&= \frac{1}{N} \frac{\int_{r \in R} r^{N+1} f(r) dr}{\int_{r \in R} r^{N-1} f(r) dr} A^{-1}
\end{aligned}
$$

Multivariate Normal

$$f(r) = e^{-\frac{1}{2}r^2}$$

$$\int_0^\infty r^k e^{-\frac{1}{2}r^2} dr = \int_0^\infty r^{k-1} e^{-\frac{1}{2}r^2} r dr$$

Let $u = \frac{r^2}{2}$. Then $du = rdr$.

$$\begin{aligned}
\int_0^\infty r^k e^{-\frac{1}{2}r^2} dr &= \int_0^\infty (2u)^{\frac{k-1}{2}} e^{-u} du \\
&= 2^{\frac{k-1}{2}} \int_0^\infty u^{\frac{k+1}{2}-1} e^{-u} du \\
&= 2^{\frac{k-1}{2}} \Gamma(\frac{k+1}{2})
\end{aligned}$$

Multivariate Normal

$$
\begin{aligned}
f(r) &= e^{-\frac{1}{2}r^2} \\
\int_0^\infty r^{N-1} e^{-\frac{1}{2}r^2} dr &= 2^{\frac{N-1-1}{2}} \Gamma(\frac{N-1+1}{2}) \\
c &= \frac{|A|^{\frac{1}{2}} \Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} \int_{r \in R} r^{N-1} f(r) dr} \\
&= \frac{|A|^{\frac{1}{2}} \Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} 2^{\frac{N-2}{2}} \Gamma(\frac{N}{2})} = \frac{|A|^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} \\
&= \frac{1}{(2\pi)^{\frac{N}{2}} |A|^{-\frac{1}{2}}}
\end{aligned}
$$

Multivariate Normal

$$\int_0^\infty r^k e^{-\frac{1}{2}r^2} dr = 2^{\frac{k-1}{2}} \Gamma(\frac{k+1}{2})$$

$$\begin{aligned} \frac{\int_0^\infty r^{N+1} e^{-\frac{1}{2}r^2} dr}{\int_0^\infty r^{N-1} e^{-\frac{1}{2}r^2} dr} &= \frac{2^{\frac{N+1-1}{2}} \Gamma(\frac{N+2}{2})}{2^{\frac{N-1-1}{2}} \Gamma(\frac{N}{2})} \\ &= 2 \frac{\frac{N}{2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2})} \\ &= N \end{aligned}$$

Multivariate Normal

$$
\begin{aligned}
\Sigma &= \frac{1}{N}\frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr} A^{-1} \\
&= \frac{1}{N} N A^{-1} \\
&= A^{-1} \\
c &= \frac{1}{(2\pi)^{\frac{N}{2}} |A|^{-\frac{1}{2}}} \\
&= \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}}
\end{aligned}
$$

Multivariate Pearson Type VII

Density function is $cf(\sqrt{z'Az})$ where

$$f(r) = (1 + r^2)^{-m}$$

$$c = \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} \int_{r \in R} r^{N-1} f(r) dr}$$

Mulitvariate Pearson Type VII

$$\begin{aligned}\int_0^\infty r^k f(r)dr &= \int_0^\infty r^k(1+r^2)^{-m}dr \\ &= \frac{1}{2}\int_0^\infty r^{k-1}(1+r^2)^{-m}2rdr\end{aligned}$$

Let $v = (1+r^2)^{-1}$. Then $r^2 = (1-v)/v$ and $dv = -(1+r^2)^{-2}2rdr = -v^2 2rdr$

$$\int_0^\infty r^k(1+r^2)^{-m}dr = \frac{1}{2}\int_1^0 (\frac{1-v}{v})^{\frac{k-1}{2}} v^m(-\frac{dv}{v^2})$$

Multivariate Pearson Type VII

$$
\begin{aligned}
\int_0^\infty r^k(1+r^2)^{-m}dr &= \frac{1}{2}\int_1^0 \left(\frac{1-v}{v}\right)^{\frac{k-1}{2}} v^m\left(-\frac{dv}{v^2}\right) \\
&= \frac{1}{2}\int_0^1 (1-v)^{\frac{k-1}{2}} v^{m-2-\frac{k-1}{2}} dv \\
&= \frac{1}{2}\int_0^1 (1-v)^{\frac{k+1}{2}-1} v^{m-\frac{k+1}{2}-1} dv \\
&= \frac{1}{2}\frac{\Gamma(\frac{k+1}{2})\Gamma(m-\frac{k+1}{2})}{\Gamma(m)}
\end{aligned}
$$

Multivariate Pearson Type VII

$$\begin{aligned}
\int_0^\infty r^k (1+r^2)^{-m} dr &= \frac{1}{2} \frac{\Gamma(\frac{k+1}{2})\Gamma(m - \frac{k+1}{2})}{\Gamma(m)} \\
\int_0^\infty r^{N-1} (1+r^2)^{-m} dr &= \frac{1}{2} \frac{\Gamma(\frac{N}{2})\Gamma(m - \frac{N}{2})}{\Gamma(m)} \\
\int_0^\infty r^{N+1} (1+r^2)^{-m} dr &= \frac{1}{2} \frac{\Gamma(\frac{N+2}{2})\Gamma(m - \frac{N+2}{2})}{\Gamma(m)}
\end{aligned}$$

Multivariate Pearson Type VII

$$
\begin{aligned}
c &= \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}\int_{r\in R} r^{N-1} f(r)dr} \\
&= \frac{|A|^{\frac{1}{2}}\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}\frac{1}{2}\frac{\Gamma(\frac{N}{2})\Gamma(m-\frac{N}{2})}{\Gamma(m)}} \\
&= \frac{|A|^{\frac{1}{2}}\Gamma(m)}{\pi^{\frac{N}{2}}\Gamma(m-\frac{N}{2})}
\end{aligned}
$$

Multivariate Pearson Type VII

$$
\begin{aligned}
\Sigma &= \frac{1}{N}\frac{\int_{r\in R} r^{N+1} f(r)dr}{\int_{r\in R} r^{N-1} f(r)dr} A^{-1} \\
&= \frac{1}{N}\frac{\frac{1}{2}\frac{\Gamma(\frac{N+2}{2})\Gamma(m-\frac{N+2}{2})}{\Gamma(m)}}{\frac{1}{2}\frac{\Gamma(\frac{N}{2})\Gamma(m-\frac{N}{2})}{\Gamma(m)}} A^{-1} \\
&= \frac{1}{N}\frac{\Gamma(\frac{N+2}{2})\Gamma(m-\frac{N+2}{2})}{\Gamma(\frac{N}{2})\Gamma(m-\frac{N}{2})} A^{-1} \\
&= \frac{1}{N}\frac{\frac{N}{2}\Gamma(\frac{N}{2})\Gamma(m-\frac{N+2}{2})}{\Gamma(\frac{N}{2})(m-\frac{N+2}{2})\Gamma(m-\frac{N+2}{2})} A^{-1} = \frac{1}{2}\frac{1}{2m-(N+2)} A^{-1}
\end{aligned}
$$