MODEL-BASED MORPHOLOGY: SIMPLE AND COMPLEX SHAPES

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1. INTRODUCTION

Filtering by morphological operations is particularly suited for removing clutter and noise objects which have been introduced into noiseless binary images containing shapes of interest. Morphological filtering is designed to exploit differences in the spatial nature (shape, size, orientation) of the objects (connected components) in the ideal noiseless images as compared to the noise/clutter objects. Since the typical noise models (union, intersection set difference, etc.) for binary images are not additive, the morphological processing is strongly nonlinear, optimal filtering results conventionally available for linear processing in the presence of additive noise are not directly applicable to morphological filtering of binary images. After morphological filtering for clutter reduction recognition can begin.

In this paper we describe a morphological filtering analog to the classic Wiener filter, a preliminary account having been given in [1] and we finish our discussion with a sketch of the morphological recognition process. The discussion begins in Section 2 with a review of the Wiener filter and its extension to a Binary Wiener filter; in these the underlying model entails decomposing the signal and additive noise into spectral elements in terms of an orthogonal basis set. Classic Wiener optimal estimation weights the respective spectral elements in the noisy signal according to the expected values of signal and noise energy across the spectrum. Section 3 extracts the essence of the algebraic structure underlying the derivation of the Wiener filter, doing so in a way that retains the concepts of energy and spectral decomposition, but eliminates the assumptions of noise additivity, orthogonal bases, and even the concept of inner product. The stage is thus set for the subsequent morphological filtering results where those assumptions do not apply. Section 4 derives an optimal morphological filter for binary images composed of the union (not sum) of the signal and noise connected components. The spectral decomposition of signal and noise is in terms of an ordered basis of connected components where the ordering is based on the morphological opening operation. (Such a basis is, in a certain sense, a "nested" collection of sets.) Thus the underlying model is based upon that ordered basis (which provides prototypes of signal and noise objects scattered throughout the binary image) and upon a morphological spectrum derived from openings. Section 5 expands the results of Section 4 beyond allowing signal and noise objects to be taken from a single ordered basis (e.g. an ordered set of discs). In Section 5, the collection of prototypes can include any number of coordinated ordered bases (e.g. an ordered set of discs, as well as an ordered set of squares, as well as several ordered sets of lines each at different orientations.) In Section 6, we give a sketch of the morphological recognition process for complex shapes.

2. THE WIENER FILTER

Regarding the discrete Wiener filter, let $b_1, \ldots, b_n$ be an orthonormal basis. The model for the ideal random signal f is that $f = \sum_{n=1}^{N} \alpha_n b_n$ where $E[\alpha_n] = 0, V[\alpha_n] = \sigma^2_{f_n}$, and $E[\alpha_m \alpha_n] = 0, m \neq n$.

Visual Form, Edited by C. Arcelli *et al.*
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The variances $\sigma^2_{f_n}$ are taken to be known. The model for the random noise g is that $g = \sum_{n=1}^{N} \beta_n$ where $E[\beta_n] = 0, V[\beta_n] = \sigma^2_{g_n}$, and $E[\beta_m \beta_n] = 0, m \neq n$. Noise and signal are uncorrelated so th $E[\alpha_n \beta_m] = 0$.

The observed noisy signal is $f + g = \sum_{n=1}^{N} (\alpha_n + \beta_n) b_n$. The Wiener filtering problem to determine weights $w_1, \ldots, w_N$ to make the estimate $\hat{f}$ of f, $\hat{f} = \sum_{n=1}^{N} w_n(\alpha_n + \beta_n) b_n$ minimi $E[\rho(f, \hat{f})]$, where ρ is a metric. In the case of Euclidean distance for the metric ρ, $E[\rho(f, \hat{f})]$ $E[\|f - \hat{f}\|^2]$.

Now,

$$\begin{aligned} \|f - \hat{f}\|^2 &= \|\sum_{n=1}^{N} \alpha_n b_n - \sum_{n=1}^{N} w_n(\alpha_n + \beta_n) b_n\|^2 \\ &= \sum_{n=1}^{N} [w_n(\alpha_n + \beta_n) - \alpha_n]^2 \end{aligned} \tag{1}$$

And

$$\begin{aligned} E\left[\|f - \hat{f}\|^2\right] &= \sum_{n=1}^{N} E[(w_n(\alpha_n + \beta_n) - \alpha_n)^2] \\ &= \sum_{n=1}^{N} w_n^2(\sigma^2_{f_n} + \sigma^2_{g_n}) - 2w_n\sigma^2_{f_n} + \sigma^2_{f_n} \end{aligned} \tag{2}$$

Hence, the minimizing weights are given by

$$w_n = \frac{\sigma^2_{f_n}}{\sigma^2_{f_n} + \sigma^2_{g_n}}. \tag{3}$$

One can also define a binary Wiener filter, with weights restricted to 0 or 1. To determin the minimizing weights, we need just examine

$$w_n^2(\sigma^2_{f_n} + \sigma^2_{g_n}) - 2w_n\sigma^2_{f_n} + \sigma^2_{f_n} = \begin{cases} \sigma^2_{f_n} & \text{if } w_n = 0 \\ \sigma^2_{g_n} & \text{if } w_n = 1 \end{cases} \tag{4}$$

Hence, under the constraint that the $w_n \in \{0, 1\}$, the minimizing weights are given by

$$w_n = \begin{cases} 0 & \text{if } \sigma^2_{f_n} < \sigma^2_{g_n} \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

In this case the estimate $\hat{f} = \sum_{n \in S} (\alpha_n + \beta_n) b_n$, where $S = \{n | w_n = 1\}$. Thus the optimal binary Wiener filter retains that part of the spectrum where the expected signal energy exceeds the expected noise energy, and discards the rest.

3. OPTIMAL FILTERING IN THE GENERALIZED CASE

This section restates the binary Wiener filter results, retaining the classic algebraic structure under far less restrictive assumptions than those of Section 2. The new assumptions will in fact be consistent with the morphological filter we will develop in Section 4. Specifically we now relax the assumptions of additive noise, vector norms, inner products, and orthonormal bases, replacing them with more general assumptions on the nature of noise inclusion, distance, energy, and spectral decomposition, and the relationships between them.

Let f be any binary image in a set B of binary images and ψ be a mapping (a spectral decomposition) taking f into the N-tuple $(f_1, \ldots, f_N)$; that is $\psi : B \to B^N$. (In the case of the Wiener filter, the N-tuple $(f_1, \ldots, f_N)$ is $(\alpha_1 b_1, \ldots, \alpha_N b_N)$. Here, we incorporate into each f_n both the scalar and the basis elements.) Let ψ^{-1} be the inverse mapping re-assembling $(f_1, \ldots, f_N)$ back into f; that is $\psi^{-1} : B^N \to B$. The identity operator can be expressed as $\psi\psi^{-1}$ and $\psi^{-1}\psi$. For any two binary images f and g in B let there be defined a binary operation $<>$ such that $f <> g$ is also a binary image in B. When g is the noise, $f <> g$ corresponds to the observed noisy binary image. We require that $<>$ and ψ satisfy the relationship

$$\psi(f <> g) = (f_1 <> g_1, \ldots, f_N <> g_N). \tag{6}$$

Let ρ be the function evaluating the closeness of one image to another. Hence $\rho : B \times B \rightarrow [0,\infty)$. The function ρ must satisfy $\rho(f,h) = \sum_{n=1}^{N} \rho(f_n, h_n)$ where $\psi(f) = (f_1, \ldots, f_N)$ and $\psi(h) = (h_1, \ldots, h_N)$.

For any binary image g, we let $\#$ represent the operator which quantifies the energy in the binary image g; $\# : B \rightarrow [0,\infty)$. The operator $\#$ must satisfy $\#g = \sum_{n=1}^{N} \#g_n$, for spectral decomposition $\psi(g) = (g_1, \ldots, g_N)$. Finally, there is a relationship between ρ and $\#$: The distance between the binary image and the ideal image is just the energy in the noise image; $\rho(f <> g, f) = \#g$.

Let $w_n \in \{0,1\}, n = 1, \ldots, N$ be binary weights and let the filtered binary image have a representation $(w_1(f_1 <> g_1), \ldots, w_N(f_N <> g_N))$ where

$$w_n(f_n <> g_n) = \begin{cases} f_n <> g_n & \text{if } w_n = 1 \\ \phi & \text{if } w_n = 0 \end{cases} \tag{7}$$

and ϕ is the binary image satisfying $f <> \phi = f$. The filtered binary image $\hat{f}$ itself can then be written as

$$\hat{f} = \psi^{-1}\left(w_1(f_1 <> g_1), \ldots, w_n(f_N, <> g_N)\right). \tag{8}$$

In essence the effect of the filtering is obtained by nulling spectral content of the observed noisy binary image.

The optimal filter parameters w_n are chosen to minimize

$$\begin{aligned} E[\rho(\hat{f}, f)] &= E\left[\sum_{n=1}^{N} \rho(\hat{f}_n, f_n)\right] = \sum_{n=1}^{N} E[\rho(w_n(f_n <> g_n), f_n)] \\ &= \sum_{n=1}^{N} E\left[\begin{cases} \#g_n & \text{if } w_n = 1 \\ \#f_n & \text{if } w_n = 0 \end{cases}\right]. \end{aligned} \tag{9}$$

Hence, the best value for w_n is given by

$$w_n = \begin{cases} 0 & \text{if } E[\#f_n] < E[\#g_n] \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Then the index set S corresponding to the spectral content that will be included in the optimal filter output can be defined by $S = \{n | E[\#f_n] \geq E[\#g_n]\}$.

4. OPTIMAL BINARY MORPHOLOGICAL FILTER

To apply the foregoing algebraic filtering paradigm to mathematical morphology, we need to define the ideal random image model, the random noise model, the relationship of the observed image to the ideal random image and random noise, the formulation of representation operator ψ from morphological operators, the energy measure $\#$, and the closeness measure ρ. We begin with the representation operator ψ, which will be formulated relative to morphological opening, where the opening of binary image A by structuring element K is defined by

$$A \circ K = \bigcup \{K_x : K_x \subseteq A\} \tag{11}$$

where subscripts having names like x or y designate a translation of the set subscripted and where we assume all images are compact subsets of k-dimensional Euclidean space R^K. (See Serra [2], Haralick, Sternberg, and Zhuang [3], or Dougherty and Giardina [4,5] for the fundamental properties of the morphological opening.)

The representation operator ψ will be defined in a manner akin to the morphological granulometric pattern spectrum. To set up our definition for ψ in a way which relates to the ideal random image and noise models, we note that the opening operator has the following property: If $A = \bigcup_{i=1}^{I} A_i$, where each A_i is a connected component of A, and K is a connected structuring element, then

$$A \circ K = (\bigcup_{i=1}^{I} A_i) \circ K = \bigcup_{i=1}^{I} (A_i \circ K). \tag{12}$$

This property, that the opening of a union of connected components is the union of the openings, will be essential throughout our development. It is this property which motivates the following definition: Two sets A and B are said to not interfere with one another if and only if X, a connected component of $A \cup B$, implies that X is a connected component of A or of B but not

both. It immediately follows that if A and B do not interfere with one another and K is a connected structuring element, then

$$(A \cup B) \circ K = (A \circ K) \cup (B \circ K). \tag{13}$$

The *opening-spectrum* operator ψ will be defined in terms of a set of openings. This set of openings will be based on the structuring elements in a naturally ordered morphological basis. We define a collection $\mathcal{K}$ of structuring elements to be an *opening spectrum basis* if and only if $K \in \mathcal{K}$ implies K is connected and $K, L \in \mathcal{K}$ implies $K \circ L = K$ or $K \circ L = \phi$. A opening-spectrum basis $\mathcal{K} = \{K(1), \ldots, K(M)\}$ is *naturally ordered* if and only if $K(1) = \{0\}$ and

$$K(i) \circ K(j) = \begin{cases} K(i) & j \leq i \\ \phi & j > i. \end{cases} \tag{14}$$

A simple example of an ordered opening-spectrum basis is a set of squares of increasing size, beginning with a square of one pixel.

Now we can define the operator ψ which produces a opening-spectrum with respect to a naturally ordered opening-spectrum basis $\mathcal{K} = \{K(1), \ldots, K(M)\}$. ψ is defined by $\psi(A) = (A_1, \ldots, A_M)$ where

$$A_m = A \circ K(m) - A \circ K(m+1) \tag{15}$$

for $m = 1, \ldots, M-1, A_M = A \circ K(M)$, and $K(1) = \{0\}$. A_m is that part of A which is open under $K(m)$ but not open under $K(m+1)$, except for A_M which is A opened by $K(M)$. It follows from this definition that for $i \neq j$, $A_i \cap A_j = \phi$. This happens because

$$\begin{aligned} A_i \cap A_j &= [A \circ K(i) - A \circ K(i+1)] \cap [A \circ K(j) - A \circ K(j+1)] \\ &= [A \circ K(i) \cap A \circ K(j)] \cap [A \circ K(i+1) \cup A \circ K(j+1)]^c \\ &= [A \circ K(\max\{i,j\})] \cap [A \circ K(\min\{i+1, j+1\})] \\ &= \phi \text{ since } \max\{i,j\} \geq \min\{i+1, j+1\} \text{ for any } i \neq j, \quad i,j < M \end{aligned} \tag{16}$$

For $i < M$,

$$\begin{aligned} A_i \cap A_M &= [A \circ K(i) - A \circ K(i+1)] \cap A \circ K(M) \\ &= ([A \circ K(i) \cap [A \circ K(M)]) \cap [A \circ K(i+1)]^c \\ &= A \circ K(M) \cap [A \circ K(i+1)]^c \quad \text{since } A \circ K(i) \supseteq A \circ K(M) \\ &= \phi \quad \text{since } A \circ K(i+1) \supseteq A \circ K(M) \end{aligned} \tag{17}$$

It is easy to see that from the opening spectrum, $(A_1, \ldots, A_M)$, the original shape A can be exactly reconstructed. Consider

$$\bigcup_{m=1}^{M} A_m = [A \circ K(1) - A \circ K(2)] \cup \ldots \cup [A \circ K(M-1) - A \circ K(M)] \cup A \circ K(M) \tag{18}$$

$$\text{Since } K(i) \circ K(j) = K(i) \text{ for } i \geq j, A \circ K(j) \supseteq A \circ K(i) \text{ for } i \geq j. \tag{19}$$

Hence the sets $A = A \circ K(1), A \circ K(2), \ldots, A \circ K(M)$ are ordered in the sense that

$$A = A \circ K(1) \supseteq A \circ K(2) \supseteq \ldots \supseteq A \circ K(M) \tag{20}$$

From this it follows that for any $m \geq 2$,

$$[A \circ K(m-1) - A \circ K(m)] \cup A \circ K(m) = A \circ K(m-1) \tag{21}$$

Now by working from the right end of the union representation, taking two terms at a time, the entire union is seen to collapse to $A \circ K(1) = A$.

ψ^{-1} can then be defined by $\psi^{-1}(A_1, \ldots, A_M) = \bigcup_{m=1}^{M} A_m$. The existence of ψ^{-1} implies that the representation is unique in the sense that two different opening spectra must be associated with two different shapes and two different shapes must be associated with two different opening spectra. It implies, as well, that the representation is complete.

Next we discuss the spatial random process generation mechanism which produces binary image realizations. A spatial random process producing a set A is a non-interfering spatial Poisson process with respect to an ordered opening-spectrum basis $\mathcal{K}$ if and only if:

- For some Z, a Poisson distributed random number (with Poisson density parameter λ_A), which is the total connected component count of a binary image realization A;

- For some multinomial distributed numbers $L_1, \ldots, L_M$ with $\sum_{m=1}^{M} L_m = Z$ (with respective multinomial probabilities $p_1, \ldots, p_M$), which split the Z connected components into M subsets containing objects of the same type;
- For some randomly chosen translations $x_{mj}, m = 1, \ldots, M;\ j = 1, \ldots, L_m$;
- $A = \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}}$, where the translated structuring elements do not interfere, i.e.,

$$K(i)_{x_{ij}} \bigcap K(m)_{x_{mn}} = \begin{cases} K(i)_{x_{ij}} & \text{if } i = m \text{ and } j = n \\ \phi & \text{otherwise.} \end{cases} \tag{22}$$

From this definition of a non-interfering random process, it follows that

$$\begin{aligned} A \circ K(\lambda) &= \left(\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \right) \circ K(\lambda) \\ &= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} [K(m)_{x_{mj}} \circ K(\lambda)] \\ &= \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \end{aligned} \tag{23}$$

Moreover, if $\psi(A) = (A_1, \ldots, A_M)$, then

$$A_m = \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \tag{24}$$

for $m = 1, \ldots, M$. We interpret these results in the following manner: If A is opened by the λth basis structuring element, all components originating from "smaller" (lower-numbered) basis structuring elements are removed; the opening spectrum of A (with respect to the basis from which it was built) sorts A according to the index number of the underlying basis structuring elements, and leaves nothing out.

We consider both the ideal random image and the noise image to be generated by non-interfering random spatial processes. The observed noisy image is the union of the ideal image with a noise/clutter image. This motivates a definition of non-interfering spatial processes which here plays the role of the zero correlation between the coefficients of the image process and the coefficients of the noise process in the Wiener filter case. A random process generating realization D and a random process generating realization E are said to be non-interfering random processes if and only if D and E are always non-interfering sets for each realization.

We can now define an observed noisy image. Let A be a realization of a non-interfering spatial process (with respect to an ordered opening-spectrum basis $\mathcal{K}$) producing images of interest and let N be a realization of a non-interfering spatial process (with respect to the same $\mathcal{K}$) producing noise/clutter. We suppose that these processes do not interfere with one another. The observed noisy realization is defined as $A \cup N$. Let $\psi(A) = (A_1, \ldots, A_M), \psi(N) = (N_1, \ldots, N_M)$, and $\psi(A \cup N) = (B_1, \ldots, B_M)$. Then

$$B_m = (A \cup N) \circ K(m) - (A \cup N) \circ K(m+1), \tag{25}$$

for $m = 1, \ldots, M-1$, and

$$B_M = (A \cup N) \circ K(M). \tag{26}$$

Because the processes do not interfere with one another,

$$\begin{aligned} B_m &= [A \circ K(m) \cup N \circ K(m)] - [A \circ K(m+1) \cup N \circ K(m+1)] \\ &= [A \circ K(m) - A \circ K(m+1)] \cup [N \circ K(m) - N \circ K(m+1)] \\ &= A_m \cup N_m \\ \text{and} \quad B_M &= A \circ K(m) \bigcup B \circ K(m) \\ &= A_M \bigcup N_M \end{aligned} \tag{27}$$

Thus we have just seen that

$$\psi(A \cup N) = (A_1 \cup N_1, \ldots, A_M \cup N_M). \tag{28}$$

The filtered image $\hat{A}$ will be based on selecting the most appropriate components from the opening-spectrum of $A \cup N$. Letting S be the set of components selected, we estimate A by $\hat{A}$ where

$$\hat{A} = \bigcup_{m \in S} (A_m \cup N_m) \text{ or } \hat{A} = \bigcup_{m \in S} B_m. \tag{29}$$

Thus by choosing the form of the estimation analogously to that of the binary Wiener filter, the estimation problem becomes one of choosing an appropriate index set S.

To determine S, we must first state our error criterion. For any two sets A and $\hat{A}$, we define the closeness (non-overlap) of A to $\hat{A}$ by $\rho(A, \hat{A}) = \#[(A - \hat{A}) \cup (\hat{A} - A)]$ where $\#$ is the set counting measure (pixel count, area). Our error criterion is then

$$E[\rho(A, \hat{A})] = E\left\{\#[(A - \hat{A}) \cup (\hat{A} - A)]\right\}. \tag{30}$$

To see how to choose S to minimize $E\left\{\#[(A - \hat{A}) \cup (\hat{A} - A)]\right\}$, first note that

$$\begin{aligned} A - \hat{A} &= \bigcup_{m=1}^{M} A_m - \bigcup_{m \in S} (A_m \cup N_m) = \bigcup_{\substack{m=1 \\ m \notin S}}^{M} A_m \\ \hat{A} - A &= \bigcup_{m \in S}^{M} A_m \cup N_m - \bigcup_{m=1}^{M} A_m = \bigcup_{m \in S} N_m. \end{aligned} \tag{31}$$

Hence,

$$\begin{aligned} \rho(A, \hat{A}) &= \#[(A - \hat{A}) \cup (\hat{A} - A)] \\ &= \#(A - \hat{A}) + \#(\hat{A} - A) \\ &= \# \bigcup_{\substack{m=1 \\ m \notin S}}^{M} A_m + \# \bigcup_{m \in S} N_m \\ &= \sum_{\substack{m=1 \\ m \notin S}}^{M} \#A_m + \sum_{m \in S} \#N_m \end{aligned} \tag{32}$$

The two summations above are respectively the area of the ideal image left out, plus the noise and clutter area left in. The individual terms decompose that area by spectral content.

Now, since each spectral component is built of translates of the same basis structuring elements, and since non-interference implies mutual exclusivity,

$$\begin{aligned} \#A_m &= \# \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \\ &= \sum_{j=1}^{L_m} \#K(m)_{x_{mj}} = L_m \#K(m) \end{aligned} \tag{33}$$

so that

$$E[\#A_m] = \#K(m) p_m \lambda_A \mathcal{A} \tag{34}$$

where p_m is the multinomial probability for the ideal image process, λ_A is the Poisson density parameter of the ideal image process, and $\mathcal{A}$ is the area of the image spatial domain. Likewise, $E[\#N_m] = \#K(m) q_m \lambda_N \mathcal{A}$, where q_m is the multinomial probability for the noise process and λ_N is the Poisson density parameter of the noise process.

To determine the index set S, we then have

$$\begin{aligned} E\left\{\#[(A - \hat{A}) \cup (\hat{A} - A)]\right\} &= E\left[\sum_{m=1}^{M} \begin{cases} \#A_m & m \notin S \\ \#N_m & m \in S \end{cases}\right] \\ &= \sum_{m=1}^{M} \begin{cases} E[\#A_m] & m \notin S \\ E[\#N_m] & m \in S \end{cases} \end{aligned} \tag{35}$$

Hence, the best S is defined by

$$S = \{m | E[\#N_m] < E[\#A_m]\}, \tag{36}$$

or equivalently for the statistical assumptions made,

$$S = \{m | q_m \lambda_N < p_m \lambda_A\}. \tag{37}$$

A spectral component is retained according to the relative expectations of that component's "leave-out" of ideal image vs. "leave-in" of noise and clutter.

Figure 1 illustrates the concept of the filter. A is the ideal binary image; B is the observed noisy image. There are four structuring elements K(1), K(2), K(3), and K(4) which constitute an ordered basis. The four component images are given by

$$\begin{aligned} B1 &= B \circ K(1) - B \circ K(2) \\ B2 &= B \circ K(2) - B \circ K(3) \\ B3 &= B \circ K(3) - B \circ K(4) \\ B4 &= B \circ K(4) \end{aligned}$$

Notice that all the binary-one pixels in B1 are noise. So the index set S, which selects which components constitute the filtered image, will not contain the index 1. The component images B2 and B3 contain more ideal image than noise so indices 2 and 3 are in S. Finally, the component image B4 has more noise than ideal image. Hence index 4 is not in S. The filtered image $\hat{A}$ is then defined by $\hat{A} = B2 \cup B3$.

5. EXTENSION TO GENERALIZED (TAU-)OPENINGS

The results we have just obtained can be extended to where the opening operation is changed to a generalized opening operation. Recall that in the previous section, each basic structuring element was just a set K. In the generalized opening operation, each basic structuring element is a collection Q of sets. The generalized opening of an image I with Q is then defined by:

$$I \circ Q = \bigcup_{L' \in Q} I \circ L'. \tag{38}$$

Regarding such generalized openings, Matheron [6] calls a filter Ψ a tau-opening if it satisfies four properties: it must be (1) anti-extensive, $\Psi(A) \subseteq A$; (2) translation invariant, $\Psi(A_x) = [\Psi(A)]_x$; (3) increasing, $A \subseteq B$ implies $\Psi(A) \subseteq \Psi(B)$; and (4) idempotent, $\Psi\Psi = \Psi$. The basic Matheron representation for tau-openings is that Ψ is a tau-opening if and only if there exists a collection Q such that Ψ is defined by eq. (38). Moreover, Q is a *base* for Inv[Ψ], the invariant class of Ψ; that is, the invariants for Ψ are unions of translations of elements in Q. For an elementary opening $A \circ K$, (K) is the base. The Matheron representation is discussed by Dougherty and Giardina [4,5], the gray-scale exension is given in [5], and both Serra [7] and Ronse and Heijmans [8] give lattice extensions.

The generalization is important because of the way it extends the underlying signal and noise spatial random process generation mechanism. For example, if the structuring elements were all line segments, the structuring element collection Q could consist of multiple orientation of line segments of the same length. The corresponding spatial random process would place non-interfering line segments at different orientations on the image. Or, the spatial random process could place non-interfering line segments, disks, or squares, on the image. For each size, the corresponding structuring element collection could be: line segments of the given size at a variety of orientations, a disk of the given size, and a square of the given size.

To see how the generalized opening can be used, we illustrate the case for which each structuring element collection contains exactly two structuring elements. Let $\mathcal{K} = \{K(1), \ldots, K(M)\}$ and $\mathcal{J} = \{J(1), \ldots, J(M)\}$ be naturally ordered opening bases. Define the collection $\mathcal{Q}$ by $\mathcal{Q} = \{Q(1), \ldots, Q(M)\}$ where $Q(m) = \{K(m), J(m)\}, m = 1, \ldots, M$. To make the ordering of the collection $\mathcal{K}$ and the collection $\mathcal{J}$ compatible, we require that

$$K(i) \circ J(j) = J(i) \circ K(j) = \phi \tag{39}$$

for $j > i$. $\mathcal{Q}$ is called a *generalized opening basis.*

Now, using the generalized opening operator, consider

$$\begin{aligned} K(i) \circ Q(j) &= K(i) \circ K(j) \cup K(i) \circ J(j) \\ &= \begin{cases} K(i) & i \geq j \\ \phi & \text{otherwise} \end{cases} \end{aligned} \tag{40}$$

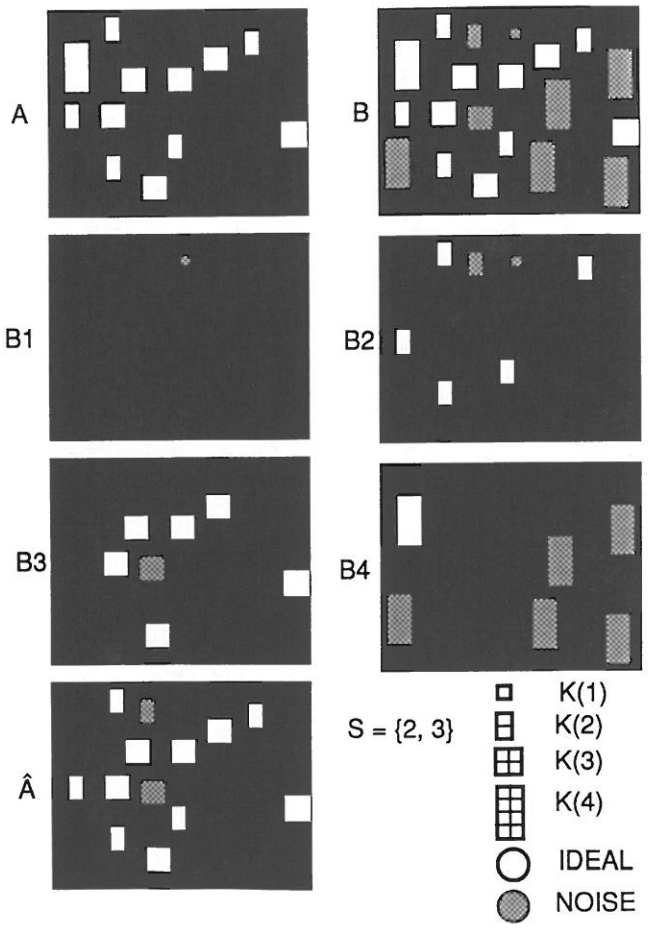

Figure 1 Figure 1 illustrates the filtering process. A is the ideal image; B is the observed noisy image. Using structuring elements K(1), K(2), K(3), and K(4) as the ordered basis produces component images B1, B2, B3, and B4. Component images B2 and B3 have more ideal image than noise, so the filtered image $\hat{A}$ is $B2 \cup B3$.

Likewise,

$$J(i) \circ Q(j) = J(i) \circ K(j) \cup J(i) \circ J(j)$$
$$= \begin{cases} J(i) & i \geq j \\ \phi & \text{otherwise} \end{cases} \tag{41}$$

Suppose that a realization A for a non-interfering process can be written as

$$A = \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \right] \bigcup \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}} \right] \tag{42}$$

where the sets in the collection

$$\{K(m)_{x_{mj}}, J(m)_{y_{mj}} : i = 1, \ldots, L_m^K, j = 1, \ldots, L_m^J\}_{m=1}^{M} \tag{43}$$

are naturally non-interfering. Then

$$\begin{aligned} A \circ Q(\lambda) &= \left[\bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \bigcup \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}} \right] \circ Q(\lambda) \\ &= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^K} [K(m)_{x_{mj}} \circ Q(\lambda)] \bigcup \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_j^J} [J(m)_{x_{mj}} \circ Q(\lambda)] \\ &= \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_j^K} K(m)_{x_{mj}} \bigcup \bigcup_{m=\lambda}^{M} \bigcup_{j=1}^{L_j^J} J(m)_{y_{mj}} \end{aligned} \tag{44}$$

Moreover, applying the spectrum definition of eq. (15) to the generalized opening Q yields

$$\begin{aligned} A_m &= A \circ Q(m) - A \circ Q(m+1) \\ &= \bigcup_{n=m}^{M} \bigcup_{j=1}^{L_j^K} K(n)_{x_{nj}} \bigcup \bigcup_{n=m}^{M} \bigcup_{j=1}^{L_j^J} J(n)_{y_{nj}} \\ &\quad - \bigcup_{n=m+1}^{M} \bigcup_{j=1}^{L_j^K} K(n)_{x_{nj}} \bigcup \bigcup_{n=m+1}^{M} \bigcup_{j=1}^{L_j^J} J(n)_{y_{nj}} \\ &= \bigcup_{j=1}^{L_j^K} K(m)_{x_{nj}} \bigcup \bigcup_{j=1}^{L_j^J} J(m)_{y_{nj}} \end{aligned} \tag{45}$$

From this it is clear that the representation operator ψ based on Q has an inverse and $A = \bigcup_{m=1}^{M} A_m$. Furthermore, $A_i \bigcup A_j = \phi$ and $\#A = \sum_{m=1}^{M} \#A_m$. This fulfills the required conditions described in Section 3. Furthermore, results for Q containing collections of pairs of structuring elements are immediately generalizable to collections having any number of structuring elements.

To extend the optimal index set S given by eq. (28) to the situation where Q contains pairs, $Q(m) = \{K(m), J(m)\}$, we need only recognize that there are now four noninterferring processes to consider: (1) a signal process involving $\{K(m)\}$ with Poisson parameter λ_{AK} and multinomial probabilities p_{Km}, (2) a signal process involving $\{J(m)\}$ with Poisson parameter λ_{AJ} and multinomial probabilities p_{Jm}, (3) a noise process involving $\{K(m)\}$ with Poisson parameter λ_{NK} and multinomial probabilities q_{Km}, and (4) a noise process involving $\{J(m)\}$ with Poisson parameter λ_{NJ} and multinomial probabilities q_{Jm}. Since eq. (35) still applies, eq. (45) applied to both signal and noise yields

$$E[p(A, \hat{A})] = \sum_{m \notin S} A\#K(m)[\lambda_{AK} p_{Km} + \lambda_{AJ} p_{Jm}] + \sum_{m \in S} A\#K(m)[\lambda_{NK} q_{Km} + \lambda_{NJ} q_{Jm}] \tag{46}$$

Thus, the best S is defined by

$$S = \{m : \lambda_{NK} q_{Km} + \lambda_{NJ} q_{Jm} < \lambda_{AK} p_{Km} + \lambda_{AJ} p_{Jm} \tag{47}$$

Extension to more than two structuring-element opening bases is straightforward.

6. RECOGNITION OF COMPLEX SHAPES

A complex shape can be considered as a union of simpler shapes. So suppose that a complex shape E has pieces $F_1, \ldots, F_N$. That is, $E = \bigcup_{n=1}^{N} F_n$. Furthermore suppose that for some subset of pieces, say $G_1, \ldots, G_M$, G_i is disconnected from G_j for $i \neq j$. In this case, we can work with the subset $\bigcup_{m=1}^{M} G_m$ of E and recognize it.

The recognition of $\bigcup_{m=1}^{M} G_m$ of connected components $G_1, \ldots, G_M$ where each G_m is some region in a component of the representation $\phi(E)$ and is in a fixed relative position to each of the other G_i's introduces an additional degree of constraint which the recognition algorithm may use. Suppose that the noise is additive and there exists structuring elements K_{ij} satisfying

$$G_i \subseteq G_j \oplus K_{ij} \quad i \neq j$$

Here the dilation can be thought of as expanding G_j just enough so that it is bigger than G_i and then translating it so that the translation, which has been incorporated in K_{ij}, just makes $G_j \oplus K_{ij}$ cover G_i.

If the observed G_i is $\hat{G}_i$ where $\hat{G}_i = G_i \cup N_i$ it makes sense to consider reducing the detected candidate $\hat{G}_i$ by the following iterative equation.

$$\hat{G}_i^0 = \hat{G}_i$$
$$\hat{G}_i^{t+1} = (\hat{G}_j^t \oplus K_{ij}) \cap \hat{G}_i^t$$

In this case, the iterative equations further reduce the candidate components. At the fixed point of the iteration, each $\hat{G}_i$ belongs to a confguration of other regions which stand in the right relationship to each other. Collecting together the configuration can then be done by locating the centroids of each of the regions and grouping those which stand in the correct relative translations with respect to one another.

7. CONCLUSION

For the problem of filtering corrupted binary images of the form $A \cup N$, we have chosen an appropriate morphological opening spectral decomposition, as well as distance and energy measures resulting in an appropriate measure of estimation error. Based upon these choices (which are quite different from the analogous choices for the additive noise/linear filter problem, and which eliminate the requirement for orthogonality or an inner product space) we have derived optimal filtering results analogous to conventional Weiner filtering results based on image and noise energy contents in each spectral bin.

The assumptions on the image and noise models in order for the results to be valid are presently fairly strong. The image and noise connected components are modeled as translated copies of objects from a single ordered opening basis set (Sections 4 and 6) or a collection of such basis sets (Sections 5 and 7). In addition there is a non-interference (non-overlap) condition so that all objects remain distinct and no objects are created that fail to arise directly from basis sets.

These conditions guarantee sufficiency. However, they are actually stronger than need be. They were sufficient to guarantee that $(A \cup B) \circ K = A \cup B$ and $\#(A \cup B) = \#A + \#B$. There are many instances in which $(A \cup B) \circ K = A \cup B$ and A and B are not non-interfering sets. If A and B are not exclusive then $\#(A \cup B) \leq \#A + \#B$. So if the sets overlap, the quantities we have been computing will be strict upper bounds. However, in this case, the overlapping can be regarded as a random process and instead of computing $\#(A \cup B)$ a composition of $E[\#(A \cup B)] = k(\#A + \#B)$ for an appropriate $0 < k < 1$ can be made. Therefore, the possibility of generalizing the results is quite strong.

Finally, we have discussed a morphological relaxation procedure to aid in the recognition of a shape from some of its components which have been detected by the morphological opening representation.

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