TWO VIEW MOTION ANALYSIS

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ABSTRACT

Given a set of corresponding points from a moving object in two perspective images, this paper gives a complete theoretical and algorithmic solution to the two view motion problem. The experimentation on the computer verifies the theory and the algorithm very well. The algorithm could be adjusted to the noise case. However, it needs an elaborate perturbation analysis.

I. Introduction

It is well known that the two view motion analysis has a special and basic importance in robotic vision. Lots of contributions have been made in recent years. Unfortunately, most research neglects, among other things, the importance of exploring the general solution of Two View - Motion Equation and differentiating modes of motion. It leads to imperfect or even incorrect results. Without a satisfactory theory it is impossible to develop a sound algorithm.

This paper briefly reviews results obtained recently by authors which comprise a complete solution to the two view motion problem. Besides, a noise robust algorithm which works well under small perturbations is included.

II. General Solution of Basic Two View - Motion Equation. Surface Assumption

Assume that a rigid body is in motion in the half-space \( z < 0 \). Take a particular point \( p \) on the object. Let \((x, y, z)\) be the spatial coordinates of \( p \) before the motion and \((x', y', z')\) be the coordinates after the motion. Let \((X, Y)\) be central projective coordinates of \( p \) before the motion and \((X', Y')\) after the motion onto the plane \( z = 1 \) with the projective center at origin 0. The following projection equations relate the 3-D point coordinates and corresponding 2-D projective point coordinates:

\[
X = x/z, \quad Y = y/z
\]

\[
X' = x'/z', \quad Y' = y'/z'
\]

As known, any 3-D rigid body motion \( M \) is equivalent to a rotation \( R \) followed by a translation \( T \) such that

\[
(x', y', z')^T = R_0 (x, y, z)^T + T_0
\]

where \( R_0 \) is a 3x3 orthonormal matrix with \( \det(R_0) = 1 \) and \( T_0 \) is 3x1 vector. 't' represents the transpo-

sition operation. From the motion equation it is easy to obtain

\[
T_0 x (X', Y', 1)^T = T_0 x (R_0 (X, Y, 1)^T)
\]

and hence the following Two View - Motion Equation

\[
(X', Y', 1) \{ T_0 x (R_0 (X, Y, 1)^T) \} = 0 \tag{1}
\]

Let

\[
T_0 = (\Delta x_0, \Delta y_0, \Delta z_0)^T
\]

\[
G_0 = \begin{bmatrix}
0 & -\Delta z_0 & \Delta y_0 \\
\Delta z_0 & 0 & -\Delta x_0 \\
-\Delta y_0 & \Delta x_0 & 0
\end{bmatrix}
\]

It is easy to verify that for any 3x1 vector \( v \) there holds

\[
T_0 x v = G_0 v
\]

As a result, Two View - Motion Equation turns out to be a familiar form

\[
(X', Y', 1) G_0 R_0 (X, Y, 1)^T = 0
\]

which is established in Tsai and Huang and Louranguini. Let

\[
E_0 = G_0 R_0
\]

Now it is clear that for any projection point correspondence pair \((X, Y, Z), (X', Y', Z')\) the following equality always holds:

\[
(X', Y', 1) E_0 (X, Y, 1)^T = 0
\]

Conversely, if a real 3x3 matrix \( E \) satisfies

\[
(X', Y', 1) E (X, Y, 1)^T = 0 \tag{2}
\]

for a set of projection point correspondence pairs, denoted by \( P \), then we have (see Zhuang and Haralick) Theorem 1.

\[
E = \alpha F_0 (\text{any real number}) \quad \text{when} \quad T_0 \neq 0 \quad \text{or} \quad G_0 \quad (G \text{any skew symmetric matrix}) \quad \text{when} \quad T_0 = 0
\]

If and only if Surface Assumption holds, that is, the surface patch or the group of surface points \( S \) which produces \( P \) cannot be contained in a quadratic surface of form

\[
(x, y, z)^T (x, y, z)^T + vt(x, y, z)^T = 0
\]
with \(|U + U^T| + |v_i| \neq 0\) and \(T^R_0 U = v^2\).

Let 
\[ A = (XX', YY', XX', XX', YY', YY', XX, YY, YX, YX, YX) \]
\[ W = \sum A_k A (k > 0, \text{ as easily seen}) \]
\[ n = (n_h, n_h, \ldots, n_h)^t \]
\[ E = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \]

Then, it is proved (see Zhuang and Haralick) that \(E\) satisfies (2) for \([(X,Y),(X',Y')] \in F\) if and only if \(h\) satisfies with
\[ W h = 0 \]  
(3)

**Theorem 2.** (see Zhuang and Haralick) Under Surface Assumption, there are exactly 8 linear independent \(A_i\)'s denoted by \(A_1, \ldots, A_8\), such that

\[ \text{Rank } (W) = \text{Rank } (\sum A_i^T A_i) = 8 \]

and hence the general solution \(h\) lies in a one dimensional subspace if and only if \(T_0 \neq 0\). There are exactly 6 linear independent \(A_i\)'s denoted by \(A_1, \ldots, A_6\) such that

\[ \text{Rank } (W) = \text{Rank } (\sum A_i^T A_i) = 6 \]

and the general solution \(h\) lies in a three dimensional subspace if and only if \(T_0 = 0\).

As a result of Theorem 2 at least 6 (8) point pairs must be contained in \(F\) depending on whether or not \(T_0\) is zero. More projection point correspondence pairs are preferable to guarantee Surface Assumption and to smooth out any noise effects. The general solution \(h\) (or the same, \(E\)) of (3) has two and only two decompositions (see Tsai and Huang, Zhuang and Haralick):

\[ E = T x R = (-T) x R' \]  
(4)

where both \(R\) and \(R'\) are orthonormal matrices of the kind. One of them equals \(R_0\), and depending on whether or not \(T_0\) is zero \(T\) is any real vector or equals \(\bar{R}_0\) with a any real number. (Note: \(T x R = [T x R_1, T x R_2, T x R_3]\). A direct procedure for decomposing \(E\) is given in Zhuang and Haralick and Section IV.

III. Determination of Mode of Motion, Rotation, Translation Orientation and Relative Depth

**Theorem 3 (Mode of Motion).** (see Zhuang and Haralick). Assume that

\[ M = (T_0, R_0) \]
and
\[ E = T x R = (-T) x R' \]
is a non-zero solution of Two View Motion Equation. Then, \((T_0, R_0) = (0, R)\) holds if and only if for any two projection point correspondence pairs \([(X_i, Y_i), (X'_i, Y'_i)] (i=1,2)\) there hold

\[ \frac{v'_i}{|v'_i|} = \frac{R v_i}{|v_i|} = 0 \]  
(5)

where 
\[ v'_i = (X'_i, Y'_i, 1)^t \]
and 
\[ v_i = (X_i, Y_i, 1)^t \]

**Theorem 4 (Rotation and Translation Orientation when \(T_0 \neq 0\))** (see Zhuang and Haralick). Assume that 
\[ M = (T_0, R_0) \]
with 
\[ T_0 \neq 0 \]
and 
\[ E(\pm T x R) = (-T) x R' \]
is a non-zero solution of Two View Motion Equation. Then,
\[ R_0 = R \]
and 
\[ T_0/|T| = \pm T/|T| \]
hold if and only if for any two projection point correspondence pairs
\[ [(X_i, Y_i), (X'_i, Y'_i)] (i=1,2) \]
there holds

\[ \frac{|v'_i|}{|v_i|} = |R v_i|/|v_i| = |T x R v_i|/|v_i| = 0 \]  
(6)

**Theorem 5.** (Relative Depth) (see Zhuang and Haralick). Assume that

\[ M = (T_0, R_0) \]
and
\[ E(\pm T x R) = (-T) x R' \]
is a non-zero solution of the Two View Motion Equation. Then the relative depth is given by:

\[ z'/z = |v_/|/|v'_i| \]  
when \(T_0 \neq 0\)

\[ z'/z = |T x R_0 v_/|/|T x R_0 v'_i| \]  
when \(T_0 = 0\)

Let
\[ A(v, v', R) = \sum |v'_i|/|v_i| - |v'_i|/|v_i| \]
\[ R(v, v', T, R) = \sum |T x R v_/| v_/ - |T x R v'_i| v'_i + |T x R v_i| T v_i \]

A more efficient and even stronger procedure to determine the mode of motion, the rotation and the translation orientation comes out:

**Theorem 6.** (See Zhuang and Haralick). Assume that
\[ M = (T_0, R_0) \]
and
\[ E(\pm T x R) = (-T) x R' \]
is a non-zero solution of Two View Motion Equation. Then, \((T_0, R_0) = (0, R)\) holds if and only if for any \(n \geq 2\) pairs
\[ [(X_i, Y_i), (X'_i, Y'_i)] \in F \]  
\(i=1,2, \ldots, n\)

\[ \sum_{i=1}^{n} A(v_i, v'_i, R) < \sum_{i=1}^{n} A(v_i, v'_i, R') \]
\[
\sum_{i=1}^{n} H(v_i, v'_i, T, R), \quad \sum_{i=1}^{n} H(v_i, v'_i, -T, R),
\]

(9)

\[
\sum_{i=1}^{n} H(v_i, v'_i, T, R'), \quad \sum_{i=1}^{n} H(v_i, v'_i, T, R')
\]

and \((T_0 || T_0 || R_0) = (\otimes T || T || R)\) holds if and only if

\[
\sum_{i=1}^{n} H(v_i, v'_i, T, R) < \sum_{i=1}^{n} A(v_i, v'_i, R),
\]

\[
\sum_{i=1}^{n} A(v_i, v'_i, R'), \quad \sum_{i=1}^{n} H(v_i, v'_i, T, R) \quad \sum_{i=1}^{n} H(v_i, v'_i, T, R')
\]

Theorem 6 is important since, based on it, a noise robust algorithm is developed. See next section.

IV. Algorithm

Step 1. Solve \( \min h^T \theta h \) with \( ||h|| = 1 \)

Step 2. Let

\[
L_1 = [h_1, h_2, h_3]
\]

\[
L_2 = [h_u, h_v, h_w]
\]

\[
L_3 = [h_7, h_8, h_9]
\]

\[
E = \begin{bmatrix}
L_1 \\
L_2 \\
L_3
\end{bmatrix}
\]

Step 3. Let

\[
\alpha = (||L_2||^2 + ||L_3||^2 - ||L_1||^2)/2
\]

\[
\beta = (||L_2||^2 + ||L_1||^2 - ||L_2||^2)/2
\]

\[
\gamma = (||L_1||^2 + ||L_2||^2 - ||L_3||^2)/2
\]

Step 4. If \( |\alpha| \geq |\beta|, |\beta| \geq |\gamma| \), then let

\[
T = \begin{bmatrix}
-L_1 & -L_2 & -L_3
\end{bmatrix}
\]

\[
T' = \begin{bmatrix}
-L_1 & -L_2 & -L_3
\end{bmatrix}
\]

\[
r_1 = [(L_2 x L_3) x L_1 + \sqrt{\alpha} (L_2 x L_3)] / ||L_1||^2 \sqrt{\alpha}
\]

\[
r_1' = [(L_2 x L_3) x L_1 - \sqrt{\alpha} (L_2 x L_3)] / ||L_1||^2 \sqrt{\alpha}
\]

\[
r_2 = (r_3 x L_2) / \sqrt{\alpha}
\]

\[
r_2' = -(r_3 x L_2) / \sqrt{\alpha}
\]

\[
r_3 = (r_3 x L_3) / \sqrt{\alpha}
\]

\[
r_3' = -(r_3 x L_3) / \sqrt{\alpha}
\]

GO TO STEP 7

(Where \( \langle ., \rangle \) represents the scalar product between two row vectors. For convenience the cross product operation 'x' also acts on two row vectors and produces a row vector.)

Step 5. If \( |\beta| \geq |\gamma| \), then let

\[
T = \begin{bmatrix}
-L_2 & -L_3 & -L_1
\end{bmatrix}
\]

\[
r_2 = [(L_2 x L_3) x L_1 + \sqrt{\beta} (L_2 x L_3)] / ||L_1||^2 \sqrt{\beta}
\]

\[
r_2' = [(L_2 x L_3) x L_1 - \sqrt{\beta} (L_2 x L_3)] / ||L_1||^2 \sqrt{\beta}
\]

\[
r_3 = (r_2 x L_2) / \sqrt{\beta}
\]

\[
r_3' = -(r_2 x L_2) / \sqrt{\beta}
\]

GO TO STEP 7

Step 6. Let

\[
E = \begin{bmatrix}
L_1 \\
L_2 \\
L_3
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
-L_3 & -L_1 & -L_2
\end{bmatrix}
\]

\[
r_3 = [(L_1 x L_2) x L_3 + \sqrt{\gamma} (L_1 x L_2)] / ||L_1||^2 \sqrt{\gamma}
\]

\[
r_3' = [(L_1 x L_2) x L_3 - \sqrt{\gamma} (L_1 x L_2)] / ||L_1||^2 \sqrt{\gamma}
\]

\[
r_1 = (r_3 x L_1) / \sqrt{\gamma}
\]

\[
r_1' = -(r_3 x L_1) / \sqrt{\gamma}
\]

\[
r_2 = (r_3 x L_2) / \sqrt{\gamma}
\]

\[
r_2' = -(r_3 x L_2) / \sqrt{\gamma}
\]

GO TO STEP 7

Step 8. If \( \sum A(v_i, v'_i, R) < \sum A(v_i, v'_i, R') \),

\[
\sum H(v_i, v'_i, T, R), \quad \sum H(v_i, v'_i, -T, R),
\]

\[
\sum H(v_i, v'_i, -T, R'), \quad \sum H(v_i, v'_i, T, R')
\]

then \{ the rigid body motion \( M \) is a pure rotation with \( B_0 = R \) and \( z' = ||v|| / ||v'|| \};

GOTO Step 12 \}

Step 9. If \( \sum A(v_i, v'_i, R') < \sum A(v_i, v'_i, R) \),

\[
\sum H(v_i, v'_i, T, R), \quad \sum H(v_i, v'_i, -T, R),
\]

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then \( \{ \text{the rigid body motion } M \text{ is a pure rotation with} \)
\[
R_0 = R' \quad \text{and} \quad z'/z = \|v_1'/v_1\|; \quad \text{GOTO Step 12}\]

**Step 10.** If \( \{ \sum H(v_1, v_1', \perp T, R') < \sum A(v_1, v_1', R), \)
\[
\sum A(v_1, v_1', R'), \quad \sum H(v_1, v_1', \perp T, R), \]
\[
\sum H(v_1, v_1', \perp T, R'), \quad \sum H(v_1, v_1', \perp T, R') \}
\]
then \( \{ (T_0/||T_0||, R_0) = (\pm T/||T||, R') \) and \( z'/z = ||TxRv_1||/||Txv||; \)

**GOTO Step 12**

**Step 11.** If \( \{ \sum H(v_1, v_1', \perp T, R') < \sum A(v_1, v_1', R), \)
\[
\sum A(v_1, v_1', R'), \quad \sum H(v_1, v_1', \perp T, R), \]
\[
\sum H(v_1, v_1', \perp T, R'), \quad \sum H(v_1, v_1', \perp T, R') \}
\]
then \( \{ (T_0/||T_0||, R_0) = (\pm T/||T||, R') \) and \( z'/z = ||TxRv_1||/||Txv||; \)

**Step 12.** STOP

**VII. Experiments and Conclusion**

**Experiment 1.**
\[ T_0 = (1, 1, 1)^T \]
\[ R_0 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Taking some eight points which obey the Surface Assumption, the computed motion parameters are as follows:
\[ T = (0.4082481, 0.4082484, 0.4082483)^T \]
\[ R = \begin{bmatrix} 0.7071066 & 0.2357018 & 0.6666666 \\ -0.2357018 & 0.2357024 & 0.6666670 \\ 0.9428092 & 0.3333333 \end{bmatrix} \]
\[ R' = \begin{bmatrix} 0.7071066 & 0.7071066 & 3.6500236E-08 \\ -0.7071066 & 0.7071066 & 3.6500238E-08 \\ -1.0185831E-07 & -1.5347749E-07 & 1.0000000 \end{bmatrix} \]

Taking two pairs within eight pairs:
\[ V_1 = (0, 0, 1)^T, \]
\[ V_1' = (-1, -1, 1)^T, \]
\[ V_2 = (0, 1/\sqrt{2}, 1)^T, \]
\[ V_2' = (-2, -2, 1)^T, \]

and computing
\[ RV_1/||V_1|| = V_1'/||V_1'||, \quad i = 1, 2 \]
\[ RV_2/||V_2|| = V_2'/||V_2'||, \quad i = 1, 2 \]

we determine the mode of motion:

\[ T_0 \neq 0 \]

After that, computing Step 10 or Step 11, we determine
\[ T_0/||T_0|| = T/||T||, \]
\[ R_0 = R' \]

**Experiment 2.**
\[ T_0 = (0, 0, 0)^T \]
\[ R_0 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Taking some six points which obey the Surface Assumption, the computed motion parameters are as follows:
\[ T = (-0.3982913, 0.5648923, -0.1492004)^T \]
\[ R = \begin{bmatrix} 0.3779559 & -0.8947892 & 0.2377008 \\ -0.8318263 & -0.4409188 & -0.3371295 \\ 0.4064657 & -7.0306033E-02 & -0.9195970 \end{bmatrix} \]
\[ R' = \begin{bmatrix} 0.7071068 & 0.7071068 & 0.0000000E+00 \\ -0.7071068 & 0.7071068 & 0.0000000E+00 \\ 1.2163268E-09 & 1.2163268E-09 & 1.0000000 \end{bmatrix} \]

Taking two pairs:
\[ V_1 = (0, 0, 1)^T, \]
\[ V_1' = (0, 0, 1)^T, \]
\[ V_2 = (0, -1/\sqrt{2}, 1)^T, \]
\[ V_2' = (-1, -1, 1)^T, \]

and computing
\[ RV_1/||V_1|| = V_1'/||V_1'||, \quad i = 1, 2 \]
\[ RV_2/||V_2|| = V_2'/||V_2'||, \quad i = 1, 2 \]

we determine
\[ T_0 = 0 \]
\[ R_0 = R' \]

since \( R'V_i/||V_i|| = V_i'/||V_i'||, \quad i = 0, 1 \).

The two view-motion problem for a single rigid body in the ideal case, i.e., without noise, is completely solved. Considering the noise case, a perturbation analysis is needed. In Zhuang and Haralick (1984) and Zhuang and Haralick (to appear), the epipolar flow-motion problem for a single rigid body in both ideal and noise cases are solved.

**REFERENCES**


