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Abstract. Nicolas Caritat de Condorcet (1743–94) was an early proponent of the use of majority systems in voting procedures. In his *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité de voix* (Condorcet, 1785), he demonstrated what is now known as the 'Condorcet Jury Theorem'. This theorem states that if $n=2m+1$ jurists act independently, each with probability $p>\frac{1}{2}$ of making the correct decision, then the probability $h_{2m+1}(p)$ that the jury (deciding by majority rule) makes the correct decision increases monotonically to 1 as m increases to infinity. Condorcet argued therefore that there are situations in which it is advisable to entrust a decision to a group of individuals of lesser competence than to a single individual of greater competence. Of course, Condorcet's theorem makes the assumption of independence and homogeneity within the group—assumptions which are seldom realistic. Some generalisations of this theorem will be presented in which (a) voter competencies vary, and (b) there is a dependence between voter decisions. Furthermore, the concept of an indirect majority system will be discussed and compared with a simple majority system.

1 Introduction

A coherent system with n components which functions if and only if k or more of the components function is called a k out of n system. 1 out of n systems are called parallel systems, while n out of n systems are called series systems. A majority system is one which functions if and only if a majority of the components function. Majority systems are commonly encountered in two areas: (1) Decision theory (for example, voting systems, certain juries, committees and boards) and (2) Reliability theory (for example, the design of safety systems and circuits).

We let $X_1, \ldots, X_n$ be indicator random variables where $p_i=P[X_i=1]$, for $i=1, \ldots, n$. In decision theory, X_i indicates whether or not the ith individual makes the 'correct' decision. In an engineering system, X_i indicates whether or not the ith component functions properly .We let $S=\sum_1^n X_i$ be the random variable indicating the number of 'successes'. If the X_i are independent with identical p, then (see Mood, 1950):

$$P(S\geq k)=\sum_{i=k}^{n}\binom{n}{i}p^i(1-p)^{n-i}=\frac{n!}{(k-1)!(n-k)!}\int_0^p x^{k-1}(1-x)^{n-k}dx.$$

Hence when $n=2m+1$ is odd, the reliability of a majority system (with independent and identically distributed components) is given by:

$$\begin{aligned}P(S\geq m+1)&=h_n(p)=h_{2m+1}(p)\\&=\sum_{i=m+1}^{2m+1}\binom{2m+1}{i}p^i(1-p)^{2m+1-i}\\&=(2m+1)\binom{2m}{m}\int_0^p x^m(1-x)^m dx.\end{aligned}$$

For $n>1$, $h_n(p)$ is convex increasing on $[0, \frac{1}{2}]$, and concave increasing on $[\frac{1}{2}, 1]$. It is also easy to see that

$$h_n(p)=1-h_n(1-p)$$

and hence in particular that $h_n(\frac{1}{2})=\frac{1}{2}$. Since if $n=2m$, $P(S>m)+\frac{1}{2}P(S=m)=h_{2m-1}(p)$, it is reasonable to confine our attention to the case when the number of components n is odd.

Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet (1743–94) was an early proponent of the use of probability theory and 'majority systems' in social and political organisation. He was a mathematician, a 'philosophe' (the last of the 'savants'?) and a politician. He served as permanent secretary of the Academy of Science from 1776 until the Academy was suppressed by the French revolution. As secretary, he was scientific confidente of Turgot, the enlightened and reformist statesman who served as Controller-General following the accession of Louis XVI in 1774. He was a contemporary of Laplace, who made great contributions to probability theory during this period (*Theorie Analytique de Probabilities* was published in Paris in 1812).

Condorcet published his first book entitled *Traite du Calcul Integral* in 1765 when he was 22. His principal work however is perhaps *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité de voix* (1785) in which he applied the calculus of probabilities to the analysis of voting and general phenomena. In *Probability and Politics: Laplace, Condorcet and Turgot*, Gillespie (1972) writes that Condorcet's interest "in electoral probabilities derived immediately from his mentor's (Turgot's) conviction that elective assemblies would better serve the purposes of national administration than the estates and other corporate bodies among which French society was still partitioned in the late 18th century aftermath of feudalism". In asking what voting procedure yields the candidate (or proposition) most likely to be the best or correct one, Condorcet would have argued in favour of simple majority rule (when there are exactly two alternatives, and we assume that individual voters act independently and moreover are correct in their judgements more than half of the time). Although a proponent of social and political change, Condorcet died at the hands of the revolution under mysterious circumstances in 1794. The following theorem, however, has been named in his honour (see Grofman & Owen, 1986 or Miller, 1986 for interesting discussions about this theorem).

Theorem 1 (Condorcet Jury Theorem)

Let $n=2m+1$ be the number of individuals in a jury or decision-making body, and p=probability that an individual makes the correct decision. We let $h_n(p)$ be the probability that a majority of individuals make the correct decision, where individuals act independently. Then if $p>\frac{1}{2}$ and $n\geq 3$,

(a) $h_n(p)>p$

and

(b) $h_n(p)\uparrow 1$ as $n\rightarrow\infty$.

To put this theorem in a historical perspective with respect to the Central Limit Theorem, Heyde (1983) writes in the *Encyclopedia of Statistical Sciences* (p. 651, volume 4) that "the result was first established for the case of Bernoulli trials [X_i, $i=1,2,\ldots$, independent and identically distributed (i.i.d. with $\Pr(X_i=1)=p$, $\Pr(X_i=0)=1-p$, $0<p<1$)]. The case $p=\frac{1}{2}$ was treated by de Moivre in 1718 and the case of general p by Laplace in 1812".

2 Condorcet Jury Theorems for heterogeneous groups

It is natural to ask about the situation when voter competencies (or component reliabilities) in a group vary (are not homogeneous). Hoeffding (1956) proved an important result stating that if S is the number of successes in n independent trials where p_i is the probability of success on the ith trial, then:

Theorem 2 (Hoeffding)

If c is a positive integer such that

$$\bar{p}=(p_1+\ldots+p_n)/n\geq\frac{c}{n}$$

then

$$P(S\geq c)\geq\sum_{i=c}^{n}\binom{n}{i}(\bar{p})^i(1-\bar{p})^{n-i}$$

Therefore, using $c=m+1$ when $n=2m+1$, one has the following generalisation of the Condorcet Jury Theorem:

Theorem 3

If $n\geq 3$ and

$$\bar{p}=\text{average voter competency}\geq\frac{1}{2}+\frac{1}{2n}$$

then

(a) $h_n(\mathbf{p})=h_n(p_1,\ldots,p_n)>\bar{p}$

and for fixed $\bar{p}$,

(b) $h_n(\mathbf{p})\rightarrow 1$ as $n\rightarrow\infty$.

Condorcet, as has been indicated, made tremendous efforts to apply theoretical probability to practical social and political situations. He would probably have argued that (see Miller, 1986) "it may be entirely reasonable to entrust an important binary decision for which there is a correct decision to a group of individuals of lesser competence (e.g. a jury) than to a single individual of greater competence (e.g. a judge)".

In the 'jury' situation, the same decision is correct for all individuals. In a binary political choice situation (say in a 2-party election or referendum) individuals may have conflicting interests. In a brief description of a model developed by Miller (1986), we may assume that voters may be divided into 2 groups—those whose 'true' interests lie in one direction, and those whose 'true' interests lie in the other. Let us say that, in this situation, the electoral process 'succeeds' if the interests of the majority prevail. We use the following notation: $n=n_A+n_B$, where n_A (respectively n_B) is the number of individuals whose true interests lie in party A (party B). Without loss of generality we assume that $n_A>n_B$, that is those whose true interests lie with party A represent the majority. We let $\bar{p}_A$ and $\bar{p}_B$ be respectively the average probabilities that members of the majority and minority will cast their votes for the interests of the majority (party A). If $\bar{q}_B=1-\bar{p}_B$, then we might describe $\bar{p}_A$ and $\bar{q}_B$ as the average voter competencies in the majority and minority groups respectively. Therefore the average probability of success (party A succeeding) in the electorate as a whole is

$$\bar{p}=(n_A\bar{p}_A+n_B\bar{p}_B)/(n_A+n_B).$$

Now let $\boldsymbol{p}$ be the vector of probabilities that individuals vote for the interest of the majority, and $h_n(\boldsymbol{p})$ be the probability that the majority interests prevail. Using the result of Hoeffding, one has the following result which is another extension of the Condorcet Jury Theorem:

Theorem 4 Miller (1986)

If $n\geq 3$ and

$$\bar{p}=\frac{n_A\bar{p}_A+n_B\bar{p}_B}{n_A+n_B}\geq\frac{1}{2}+\frac{1}{2(n_A+n_B)}$$

then

(a) $h_n(\boldsymbol{p})>\bar{p}$

and for fixed $\bar{p}$

(b) $h_n(\boldsymbol{p})\to 1$ as $n=n_A+n_B\to\infty$.

In the usual case where average voter competencies ($\bar{p}_A$ and $\bar{q}_B$) both exceed $\frac{1}{2}$, we note that

$$\bar{p}\geq\frac{1}{2}+\frac{1}{2(n_A+n_B)}$$

if and only if

$$\frac{n_A}{n_B}\geq\frac{\bar{q}_B-\frac{1}{2}}{\bar{p}_A-\frac{1}{2}}+\frac{1}{2n_B(\bar{p}_A-\frac{1}{2})}.$$

Hence for large n, one may expect the 'correct' or majority party candidate to be elected if and only if

$$\frac{n_A}{n_B}>\frac{\bar{q}_B-\frac{1}{2}}{\bar{p}_A-\frac{1}{2}}.$$

This will always be the case if average competencies of voters exceed $\frac{1}{2}$ and are the same in both groups (i.e. $\bar{p}_A=\bar{q}_B$).

Now consider an example where the minority is more 'competent' on the average than the majority. In particular assume that $n_A=(0{\cdot}60)n$, $\bar{p}_A=0{\cdot}6$, $\bar{q}_B=0{\cdot}8$ and that n is large. Here $\bar{q}_B$=average competence of minority $>\bar{p}_A$=average competence of majority, and we note that:

$$\frac{n_A}{n_B}=1{\cdot}5<3=\frac{\bar{q}_B-\frac{1}{2}}{\bar{p}_A-\frac{1}{2}}.$$

One may show (using another result of Hoeffding or otherwise) that the will of the minority is likely to prevail and that in fact as $n\to\infty$ the chances of this $\to 1$.

3 Condorcet Jury Theorems modelling dependence between voters

We have discussed generalisations of the Condorcet Jury Theorem where voter competencies may vary. Another direction of generalisation is to the situation where dependence exists between the actions of the voters (or between components in a coherent system). Boland, Proschan & Tong (1989) discuss two ways of modelling dependence in majority systems. In their Model 1, it is assumed that there are $n=2m+1$ voters whose decisions are represented by the indicator random variables $Y, X_1, \ldots, X_{2m}$. Here the probability p that an individual makes the correct decision is the same for all, i.e. $p=1-q=P(Y=1)=P(X_i=1)$ for all $i=1, \ldots, 2m$. Furthermore, it is assumed that $\operatorname{Corr}(Y, X_i)=r$ for all $i=1, \ldots, 2m$ for some $r \in [0, 1]$, and that $X_1, \ldots, X_{2m}$ when conditioned on Y are independent. One may view Y as a 'voting leader' who influences the decisions of the $2m$ members $X_1, \ldots, X_{2m}$, but such that $X_1, \ldots, X_{2m}$ are independent once Y has made a decision. From the above assumptions, one may derive the following conditional probabilities:

$$
\begin{aligned}
&P(X_i=1 \mid Y=1)=p+rq \\
&P(X_i=0 \mid Y=1)=q-rq \\
&P(X_i=1 \mid Y=0)=p-rp \\
&P(X_i=0 \mid Y=0)=q+rp \qquad i=1, \ldots, 2m.
\end{aligned}
$$

One interpretation of r is that it represents the probability that the ith individual 'follows' the leader. Letting $h_n^*(p, r)$ be the probability that a majority in the group makes the correct decision in this situation, they show that:

Theorem 5 (Boland, Proschan & Tong (1989))

If $n \geq 3$ and $p > \frac{1}{2}$, then

(a) $h_n^*(p, r)$ is a decreasing function of r and $h_n^*(p, r) > p$ for $r < 1$

and

(b) $h_n^*(p, r) \to 1$ as $n \to \infty$ whenever $r < 1 - \dfrac{1}{2p}$.

An outline proof of this result is given in the Appendix. Figure 1 gives graphs of $h_{15}^*(p, r)$ for various values of r. When $r=0$, $h_n^*(p, r)=h_n(p)$, and hence $h_{15}^*(p, 0)$ represents a graph of the probability that a majority prevails in a group of 15 independent voters. Large values of r indicate a strong influence of the 'leader'.

In Model 2 of Boland, Proschan & Tong, $X_1, \ldots, X_{2m+1}$ are the indicator random variables determining the structure of a majority system of size $n=2m+1$. In this model, there is in addition an indicator random variable Y such that $p=1-q=P(Y=1)=P(X_i=1)$ and moreover $\operatorname{Corr}(Y, X_i)=r$ for $i=1, \ldots, 2m+1$. We may view Y as representing an external influence (such as an advertising or promotional campaign in the decision theoretic setting or as an indication of external conditions in the reliability setting) which indirectly affects or influences the result of the majority system. The conditional probabilities of X_i given Y are the same as in Model 1, but here Y has no direct role in determining the state of the majority system. If we let $h_n^{**}(p, r)$ be the probability that the majority makes a correct decision, then the following (an outline proof of which appears in the Appendix) may be proved:

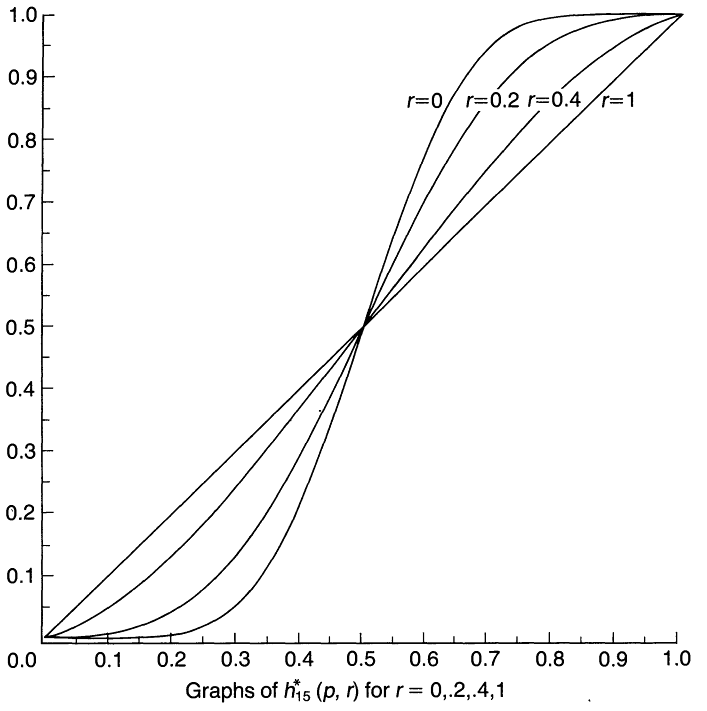


Fig. 1. Reliability of simple majority systems of size 15.

Theorem 6 (Boland, Proschan & Tong (1989))

If $n \geq 3$ and $p > \frac{1}{2}$, then

(a) $h_n^{**}(p, r)$ is a decreasing function of r and $h_n^{**}(p, r) > p$ for $r < 1$,

and

(b) $h_n^*(p, r) \rightarrow 1$ as $n \rightarrow \infty$ whenever $r < 1 - \dfrac{1}{2p}$.

4 Direct and indirect majority systems

Up to this point we have considered majority systems which are sometimes termed simple or direct. More intricate majority systems are however often usefully employed. Suppose for example that 15 individuals are to make a binary decision. We could employ a simple or direct majority criterion. Alternatively we could employ a 3×5 indirect majority criterion. Here the 15 individuals are broken into 3 groups of size 5 each. Within each subgroup of 5, a decision (1 or 0) is made by majority rule. An overall decision is then made by simple majority with respect to the 3 group decisions. Note that in a 3×5 indirect majority system, the 'correct' decision may be arrived at when as few as 6 individuals make the 'correct' decisions, but also that the 'incorrect' decision may be reached when as many as 9 individuals make the 'correct' decision.

Group 1	□	□	□	□	□
Group 2	□	□	□	□	□
Group 3	□	□	□	□	□

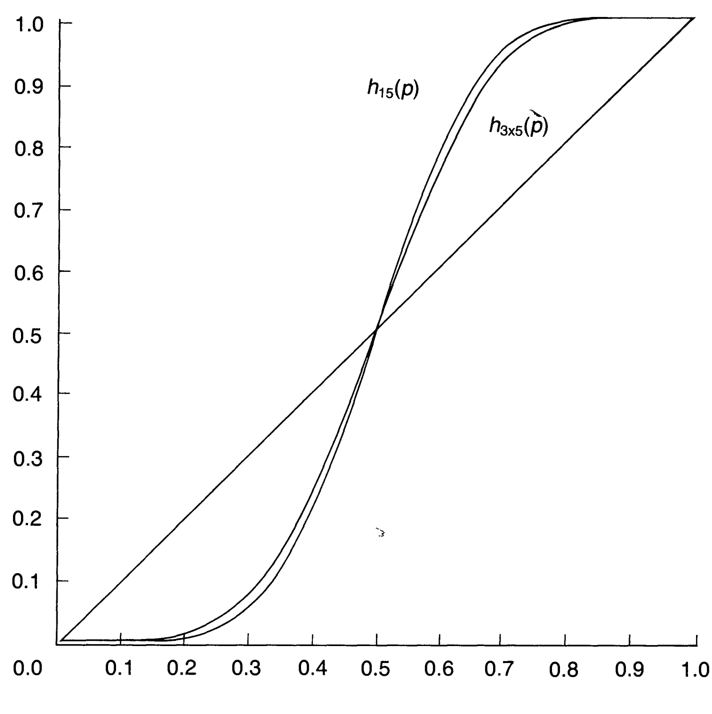
Fig. 2

More generally, for odd integers n_1, n_2 we may define an indirect $n_1 \times n_2$ majority system as one where the $n = n_1 n_2$ individuals are broken up into n_1 groups of size n_2 each. A 'correct' decision is then made if at least $(n_2 + 1)/2$ individuals in at least $(n_1 + 1)/2$ groups vote correctly. An interesting result comparing simple and indirect majority systems (Boland, Proschan & Tong) is the following (see the Appendix for an indication of the proof):

Theorem 7

Let p be the probability that any individual in a group of n makes the correct decision. Assuming voters act independently of one another, one may show that if $p > \frac{1}{2}$, then a direct majority system is always preferable to any indirect majority system of the same size.

Figure 3 compares the reliability of a simple direct majority system of size 15 with the reliability of a 3×5 indirect majority system.



Graphs of h_{3x5} (p) and h_{15} (p)

Fig. 3. 3×5 vs 1×15 indirect majority systems.

Of course, one may also consider indirect majority systems where the subgroups may be of unequal size. Indirect majority systems are in common use. For example, election of a president in the U.S.A. is basically done by an indirect majority system. Here the groups are the states (plus the District of Columbia). When restriction is made to the two main (Republican and Democratic) contenders, each group decides by majority rule to which of the two candidates it should pledge its electoral college votes (the number of electoral college votes in a group is (roughly) proportional to the size of the group). The candidate with the most electoral college votes is the winner (not necessarily the candidate receiving an overall simple majority of the original votes cast).

It may be of interest to compare two different indirect majority systems of the same overall size. For example, can we compare a 3×5 indirect majority system with a 5×3 indirect majority system? For a given p which system is more likely to give the 'correct' decision? If we let $h_{n_1\times n_2}(p)$ be the probability that an indirect $n_1\times n_2$ majority system makes the 'correct' decision, then one may in fact show quite easily that

$$h_{5\times 3}(p)\geq h_{3\times 5}(p) \Leftrightarrow p\geq \tfrac{1}{2}.$$

An interesting conjecture is that if $n_1>n_2$, then $h_{n_1\times n_2}(p)\geq h_{n_2\times n_1}(p) \Leftrightarrow p\geq \frac{1}{2}$. This would imply of course that (for $p\geq\frac{1}{2}$) it is better to have a large number of small groups than a small number of large groups in decision making.

5 Conclusions

What conclusions are we able to draw from our investigations into generalisations of the Condorcet Jury Theorem? We might be willing to say that majority systems can achieve (for a given $p>\frac{1}{2}$) arbitrarily high reliability (or high probability of making the correct decision) as the size of the system (decision body) increases, assuming dependence between voters is not too great. It also seems that for a given p, the effectiveness of a direct majority system decreases as the dependence between voters increases. Finally, indirect majority systems are (when voters act independently) not as effective as simple or direct majority systems.

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Appendix

For integers k and n where $0 \leq k \leq n$, we let

$$h_n(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Note then that for $n=2m+1$, $h_{m+1|2m+1}(p)=ph_{m|2m}(p)+qh_{m+1|2m}(p)$. For $p>\frac{1}{2}$, the Condorcet Jury Theorem implies that $h_{m+1|2m+1}(p)\uparrow 1$ as m (or n)$\rightarrow\infty$, from which it follows that $h_{m|2m}(p\rightarrow 1$ and $h_{m+1|2m}(p)\rightarrow 1$ as $m\rightarrow\infty$.

Outline proofs of theorems 5 and 6

Given $n=2m+1$ and the structure of Models 1 and 2, it is easy to verify that:

$$h_n^*(p, r)=ph_{m2m}(p+rq)+qh_{m+1|2m}(p-rp)$$

and

$$h_n^{**}(p, r)=ph_{m+1|2m+1}(p+rq)+qh_{m+1|2m+1}(p-rp).$$

One may show that

$$\frac{\delta}{\delta r}h_n^*(p, r)=mpq\binom{2m}{m}(1-r)^m[(p+rq)^{m-1}q^m-p^m(q+rp)^{m-1}],$$

and hence $h_n^*(p, r)$ is a decreasing function of r

$$\left(\frac{p}{q}\right)^m > \left(\frac{p+rq}{q+rp}\right)^{m-1} \Leftrightarrow p \text{ for all } r \in (0, 1).$$

Differentiation with respect to r shows that this is the case $\Leftrightarrow p>\frac{1}{2}$. Similarly

$$\frac{\delta}{\delta r}h_n^{**}(p, r)=(2m+1)\binom{2m}{m}pq[(p+rq)^m(q-rq)^m-(p-rp)^m(q+rp)^m]$$

which is negative for all $r \in (0, 1) \Leftrightarrow p>\frac{1}{2}$. Therefore for $p>\frac{1}{2}$, $h_n^*(p, r)$ and $h_n^{**}(p, r)$ are decreasing functions of $r \in (0, 1)$. Since $h_n^*(p, 1)=p=h_n^{**}(p, 1)$, it follows that for $p>\frac{1}{2}$, $h_n^*(p, r)>p$ and $h_n^{**}(p, r)>p$ whenever $r \in (0, 1)$.

Note now that for $p>\frac{1}{2}$,

$$r<1-\frac{1}{2p} \Leftrightarrow p-rp>\frac{1}{2}.$$

Since $h_n^*(p, r)=ph_{m|2m}(p+rq)+qh_{m+1|2m}(p-rp)$, it follows from the above remarks that as $m\rightarrow\infty$, both $h_{m|2m}(p+rq)$ and

$$h_{m+1|2m}(p-rp)\rightarrow 1 \quad \text{for } r<1-\frac{1}{2p}.$$

Hence $h_n^*(p, r)\rightarrow 1$ as $n\rightarrow\infty$. Similarly one shows that $h_n^{**}(p, r)\rightarrow 1$ as $n\rightarrow\infty$ for

$$r<1-\frac{1}{2p}.$$