Journal of Functional Analysis **165**, 258–292 (1999)
Article ID jfan.1999.3414, available online at http://www.idealibrary.com on IDEAL®

A Geometric Spectral Theory for n-tuples of Self-Adjoint Operators in Finite von Neumann Algebras*

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Communicated by A. Connes

Received July 15, 1998

Suppose $b_1, ..., b_n$ are self-adjoint elements in a finite von Neumann algebra M with trace τ and define a map Ψ from M to complex $(n+1)$-space by the formula $\Psi(x) = (\tau(x), \tau(b_1 x), ..., \tau(b_n x))$. Next let B denote the image of the positive unit ball of M under the map Ψ. B is called the *spectral scale* of $\tau, b_1, ..., b_n$. It is clearly compact and convex. The main theme of this work is that the geometry of the spectral scale B reflects spectral data for the b_i's. For example, in the finite dimensional case the operators commute if and only if the spectral scale is a polytope. Thus, one can "see" that the operators commute from the shape of spectral scale. In the case of a single operator, where the scale lies in the plane, the slopes of the boundary fill out the spectrum of the operator, corners correspond to gaps in the spectrum, and flat sports indicate eigenvalues. Analogous results hold when there is more than one operator. In the commutative setting, the spectral scale "determines" the $(n+1)$-tuple $(\tau, b_1, ..., b_n)$. However, an example is given that shows this is not generally true in the noncommutative case. Finally, a matricial version of the spectral scale is shown to be sufficient to completely determine the $(n+1)$-tuple $(\tau, b_1, ..., b_n)$. © 1999 Academic Press

Key Words: finite von Neumann algebra; spectrum; convexity.

* Each author gratefully acknowledges the support of the National Science Foundation.

0022-1236/99 $30.00

INTRODUCTION

The notion of the spectrum of an operator on Hilbert space arose in the 1920s and has played a major role in the theory of such operators ever since. In view of the obvious value of this concept it is natural to try to see if it is possible to find generalizations of this notion to n-tuples of operators and a great deal of effort has been expended in this direction.

These attempts seem to divide naturally into two avenues of approach. In the first one attempts to define a joint spectrum for *commuting* n-tuples of operators $a_1, ..., a_n$ in a Banach algebra A. Here it is natural to try to use the maximal ideal space of A to define the joint spectrum of $a_1, ..., a_n$. Although this definition is relatively easy, the resulting object depends on the ambient algebra A. This defect was repaired by Taylor in [Tay] who used homological techniques (the Koszul complex) to obtain a definition of the joint spectrum that is independent of the ambient algebra. A good deal of further work in this direction has been done. (See [Bu, CR, Vas], for example.)

Hermann Weyl initiated another approach in the late twenties while he was studying the quantization problem for the position and momentum operators. Using the Fourier transform and the one parameter unitary groups associated to these operators, he was able to obtain a functional calculus (now called the Weyl functional calculus) for this pair of noncommuting operators [We, Sect. 45]. This work was extended to arbitrary n-tuples of noncommuting self-adjoint operators by Anderson in his thesis [A]. (Also, see [N].) Work in this direction continues to this day. (See [K], for example.)

Concepts of the "joint numerical range" have also been studied. (See [Be, Ca1, Ca2, Ch], for example.) In another direction, Chandler Davis introduced his notion of the *shell* of an operator [Da1–Da3]. This is a convex compact subset of $\mathbb{R}^3$ whose geometry reflects some spectral properties of the operator.

This paper is the first of several in which an entirely new geometric approach will be employed to capture, in a single compact convex subset of real $(n+1)$ space, the spectral data for all real linear combinations of an n-tuple $b_1, ..., b_n$ of self-adjoint operators. With our present techniques we need the existence of a faithful tracial state on the C^*-algebra generated by the b_i's. Although this a significant restriction, the theory applies in many interesting cases. We expect that this new geometric approach to spectral theory (together with modern techniques of computer graphics and imaging) will give a useful procedure for computing (and "visualizing") spectral information in many cases, including, for example, in von Neumann algebras generated by the left regular representations of discrete groups. The theory also appears to dovetail nicely with Voiculescu's notions of "free probability" and "free entropy" [DNV, Voi].

Let us now describe our approach in more detail. The setting is as follows. We have an n-tuple $b_1, ..., b_n$ of self-adjoint operators that lie in a finite von Neumann algebra M equipped with the faithful normal tracial state τ. Let Ψ denote the map of M to $\mathbb{C}^{n+1}$ defined by

$$\Psi(a) = (\tau(a), \tau(b_1 a), ..., \tau(b_n a))$$

and write $B = \Psi(M_1^+)$, where $M_1^+ = \{a \in M: 0 \leqslant a \leqslant 1\}$. Since τ is normal, B is a compact convex subset of $\mathbb{R}^{n+1}$. We call B the *spectral scale* of the b's *relative to* τ (Definition 2.1).

In Section 1 we consider the special case where $n = 1$. The key result here is Lemma 1.3 which shows that the images under Ψ of certain spectral projections of $b = b_1$ lie on the boundary of the spectral scale B. Building on this result, Theorems 1.5 and 1.6 explore how spectral data for b corresponds to geometric data for the boundary of B in $\mathbb{R}^2$. For example, a real number s is an eigenvalue for b if and only if the boundary of B contains a line segment with slope s. Also, corners on the boundary correspond to gaps in the spectrum. This correspondence is summarized explicitly in Theorem 1.7.

In Section 2 we move on to the general case of n operators. Theorem 2.3 is a generalization of portions of Theorems 1.5 and 1.6 by means of which we identify spectral data for real linear combinations of the b's from the geometry of the spectral scale B. Thus, Theorem 2.3 shows how spectral data is stored in the geometry of B. Since B is convex and compact, the extreme points of B and the equations of the supporting hyperplanes of B contain all the geometric information about B. Hence they determine all of the spectral data of any real linear combination of the b's.

Let N denote the von Neumann subalgebra of M generated by $b_1, ..., b_n$ and the identity. Since we want to get information about $b_1, ..., b_n$ from the spectral scale, we must be sure that it contains no extraneous information introduced by the images under Ψ of elements of M that do not lie in N, i.e., that $\Psi(M_1^+) = \Psi(N_1^+)$ (Theorem 2.4). The assumption that τ is a trace is the key to this result.

In Section 3 we study the question of the extent to which the spectral scale "determines" the $(n+1)$-tuple $\tau, b_1, ..., b_n$. More precisely, given $(n+1)$-tuples $\tau_1, b_1, ..., b_n$ and $\tau_2, c_1, ..., c_n$ that determine the same spectral scale, can we conclude that there is an unitary transformation that intertwines the tracial representations of the von Neumann algebras that they generate? It is shown in Theorem 3.3 that the answer is yes in the case where the n-tuples are abelian. However, we show in Example 3.4 that, even in the finite dimensional, noncommutative case the answer may be no.

Thus, as is often the case in non-commutative situations, a first level object such as B may not be provide sufficient data (see, e.g., [P, EW]).

It seems necessary to introduce the *complete spectral scale* (Definition 3.5) to obtain the desired unitary equivalence. This is a sequence of spectral scales, where the nth term in the sequence is a spectral scale constructed from the tensor product of M with the $n \times n$ matrices over $\mathbb{C}$. The main result in this section (Theorem 3.11) is that the complete spectral scales are equal if and only if the von Neumann algebras that the n-tuples generate are unitarily equivalent in their tracial representations. An important component of the proof is Theorem 3.2 which shows exactly what information is carried by B in terms of the b's and the trace.

1. THE SINGLE VARIABLE CASE

In this section we begin with an analysis of the problem described in the introduction for the case $n=1$. It is convenient to begin by introducing some notation.

If M is a von Neumann algebra, we write

$$M_1^+ = \{a \in M\colon 0 \leqslant a \leqslant 1\}.$$

Also, for a self-adjoint element b in a von Neumann algebra and a real number s, we denote the spectral projection of b for the interval $(-\infty, s]$ (resp., $(-\infty, s)$) by p_s^+ (resp., p_s^-). We use $p_s^\pm$ to indicate either of these projections and we write p_s when $p_s^+ = p_s^-$. If p and q are projections in a von Neumann algebra and $p \leqslant q$ then we write

$$[p, q] = \{a\colon p \leqslant a \leqslant q\}$$

for the *order interval* that they determine.

The following is a slight restatement of Theorem 2.2 in [AP].

THEOREM 1.1. *If M and N are von Neumann algebras, Ψ is a normal linear map from M to N and F is a face in $\Psi(M_1^+)$, then there are unique projections p and q in M with $p \leqslant q$ such that*

$$\Psi^{-1}(F) \cap M_1^+ = [p, q]$$

and $F = \Psi([p, q])$.

The main results in this section are presented in Theorems 1.5 and 1.6 where it is shown how spectral data about b can be read off from the geometry of B. The implications of these results are then summarized in Theorem 1.7. This work is the basis for all subsequence material. These theorems are fundamentally grounded in Lemma 1.3 below. It is convenient to present part of the proof of this result as a separate lemma.

LEMMA 1.2. *If b is a self-adjoint element in a von Neumann algebra M, s is a real number, $c \in [p_s^-, p_s^+]$ and $a \in M_1^+$, then the following statements hold.*

(1) *We have $(b - s1)(1 - c) = (b - s1)(1 - p_s^+) \geqslant 0$ and the range projection of $(b - s1)(1 - p_s^+)$ is $1 - p_s^+$.*

(2) *If $a^{1/2}(b - s1)(1 - c)\, a^{1/2} = 0$, then $a \leqslant p_s^+$.*

(3) *We have $(s1 - b)\, c = (s1 - b)\, p_s^- \geqslant 0$ and the range projection of $(s1 - b)\, p_s^-$ is p_s^-.*

(4) *If $(1 - a)^{1/2}\, (s1 - b)\, c(1 - a)^{1/2} = 0$ then $a \geqslant p_s^-$.*

Proof. It is convenient to use the decomposition $1 = p_s^- + (p_s^+ - p_s^-) + (1 - p_s^+)$ to view the elements under consideration as matrices. Thus we have

$$b = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & s1 & 0 \\ 0 & 0 & b_2 \end{bmatrix}, \qquad \text{where} \quad p_s^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad p_s^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also, since $p_s^- \leqslant c \leqslant p_s^+$ we get

$$c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(Of course, if $p_s^- = p_s^+$, then the middle direct summands do not appear. In this case, the calculations are even more straightforward.) With this we have

$$(b - s1)(1 - c) = \begin{bmatrix} b_1 - s1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 - s1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - c' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 - s1 \end{bmatrix} = (b - s1)(1 - p_s^+).$$

Next, since p_s^+ is the spectral projection of b determined by the interval $(-\infty, s]$, we have $b(1-p_s^+) \geqslant s(1-p_s^+)$. Hence, $(b-s1)(1-c)=(b-s1)(1-p_s^+) \geqslant 0$. It is straightforward to show that the range projection of $(b-s1)(1-c)$ is $1-p_s^+$ using spectral theory.

Next write

$$a^{1/2} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12}^* & a_{22} & a_{23} \\ a_{13}^* & a_{23}^* & a_{33} \end{bmatrix}.$$

With this it is easy to check that the diagonal entries in $a^{1/2}(b-s1)(1-c)\,a^{1/2}$ are

$$a_{13}(b_2-s1)\,a_{13}^*, \qquad a_{23}(b_2-s1)\,a_{23}^*, \qquad \text{and} \qquad a_{33}(b_2-s1)\,a_{33}.$$

If $a^{1/2}(b-s1)(1-c)\,a^{1/2}=0$, then since $b_2-s1 \geqslant 0$, we get

$$a_{13}(b_2-s1)\,a_{13}^* = a_{23}(b_2-s1)\,a_{23}^* = a_{33}(b_2-s1)\,a_{33} = 0$$

and therefore

$$(b_2-s1)\,a_{13}^* = (b_2-s1)\,a_{23}^* = (b_2-s1)\,a_{33}^* = 0.$$

Since the range projection of $(b-s1)(1-p_s^+)$ is $1-p_s^+$ and $(1-p_s^+)\,a_{i3}^* = a_{i3}^*$, we get

$$a_{13}^* = a_{23}^* = a_{33}^* = 0$$

and so

$$a^{1/2} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12}^* & a_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $a=(a^{1/2})^2 \leqslant p_s^+$, as desired.

Thus, assertions (1) and (2) are true. Assertions (3) and (4) are proved using analogous arguments. ∎

LEMMA 1.3. *If M is a von Neumann algebra with a faithful finite normal tracial state τ, b is a self-adjoint element in M and s is a real number, then the following statements hold for each $c \in [p_s^-, p_s^+]$ and $a \in M_1^+$.*

(1) *If* $\tau(a)=\tau(c)$, *then* $\tau(ba)\geqslant\tau(bc)$.

(2) *If* $\tau(a)=\tau(c)$ *and* $\tau(ba)=\tau(bc)$ *then* $a\in[p_s^-, p_s^+]$; *moreover, if* $c=p_s^\pm$, *then equality in the second equation obtains only for* $a=c$.

Proof. To show that (1) holds, observe that

$$(b-s1)(1-c)=(b-s1)(1-p_s^+)\geqslant 0$$

by part (1) of Lemma 1.2 and so $b(1-c)\geqslant s(1-c)$. Since the trace is central and preserves order, we have

$$\tau(b(1-c)\,a)=\tau(a^{1/2}b(1-c)\,a^{1/2})\geqslant\tau(a^{1/2}s(1-c)\,a^{1/2})=\tau(s(1-c)\,a). \qquad (*)$$

Similarly, we have $bc\leqslant sc$ by part (3) of Lemma 1.2 and so

$$\tau(bc(1-a))\leqslant\tau(sc(1-a)). \qquad (**)$$

Using $(*)$, $(**)$, some simple algebra and the hypothesis in (1), we get

$$\begin{aligned}\tau(ba)-\tau(bc)&=\tau(b(1-c)\,a)-\tau(bc(1-a))\\&\geqslant\tau(s(1-c)\,a)-\tau(sc(1-a))\\&=\tau(sa)-\tau(sc)\\&=0.\end{aligned}$$

Thus $\tau(ba)\geqslant\tau(bc)$, as desired.

For (2), suppose $\tau(a)=\tau(c)$ and $\tau(ba)=\tau(bc)$. In this case we have

$$\tau((1-c)a)=\tau(a-ca)=\tau(c-ca)=\tau(c(1-a))$$

and

$$\tau(b(1-c)a)=\tau(ba-bca)=\tau(bc-bca)=\tau(bc(1-a)).$$

Combining these observations with $(*)$ and $(**)$ we get

$$\tau(bc(1-a))=\tau(b(1-c)\,a)\geqslant s\tau((1-c)\,a)=s\tau(c(1-a))\geqslant\tau(bc(1-a)). \qquad (***)$$

Hence all of the terms in $(***)$ are equal. In particular we have

$$\tau(b(1-c)\,a)=s\tau((1-c)\,a)$$

and

$$\tau(bc(1-a)) = s\tau(c(1-a)).$$

Hence,

$$\tau(a^{1/2}(b-s1)(1-c)\,a^{1/2}) = 0$$

and

$$\tau((1-a)^{1/2}\,(b-s1)\,c(1-a)^{1/2}) = 0.$$

Since τ is faithful we get

$$a^{1/2}(b-s1)(1-c)\,a^{1/2} = 0$$

and

$$(1-a)^{1/2}\,(b-s1)\,c(1-a)^{1/2} = 0.$$

Applying parts (2) and (4) of Lemma 1.2 we get

$$p_s^- \leqslant c \leqslant p_s^+,$$

as desired.

The final assertion in (2) follows from the fact that τ is faithful. Indeed, since a is comparable to $p_s^\pm$ and $\tau(a) = \tau(p_s^\pm)$, we must have $a = p_s^\pm$. ∎

Remark. Observe that the fact that the trace is central seems to be an essential ingredient in the proof of Lemma 1.3. It is for this reason that our results only apply to finite von Neumann algebras.

We now review the previously introduced notations for the convenience of the reader and also define some new ones.

Notation 1.4. The following notations have been previously introduced.

(1) Throughout this section M denotes a finite von Neumann algebra with a faithful finite normal tracial state τ and b stands for a self-adjoint element in M.

(2) We use Ψ to denote the map defined by the formula

$$\Psi(a) = (\tau(a), \tau(ba))$$

and we write

$$B = \Psi(M_1^+),$$

where $M_1^+ = \{a \in M: 0 \leqslant a \leqslant 1\}$.

(3) If p and q are projections in M and $p \leqslant q$, then the order interval that they determine is

$$[p, q] = \{a \in M_1^+ : p \leqslant a \leqslant q\}.$$

(4) If s is a real number, then we write p_s^+, (resp., p_s^-) for the spectral projection of b corresponding to the interval $(-\infty, s]$ (resp., $(-\infty, s)$) and we use $p_s^{\pm}$ to indicate either of these projections.

Now let us introduce some new notation.

(5) We define the *lower boundary* of B to be the set of points (x, y) in B such that if (x, y') is in B, then $y' \geqslant y$, and the *upper boundary* to be the set of points (x, y) in B such that if (x, y') is in B, then $y' \leqslant y$.

(6) We denote the *spectrum* of b by $\sigma(b)$ and the *point spectrum* of b by $\sigma_p(b)$. Also, we write s_{min} and s_{max} for the left and right endpoints of the spectrum of b, respectively.

(7) For any (x_0, x_1) on the lower boundary, write $f(x_0) = x_1$. Thus, the lower boundary is the graph of the function f. We call f the *lower boundary function* determined by B. Note that the domain of f is $[0, 1]$. As usual, we say that f is *differentiable* at 0 (resp. 1) if the right (resp. left) derivative exists at 0 (resp. 1).

(8) If s and α are real numbers, then let $L(s, \alpha)$ denote the line with the equation

$$-sx_0 + x_1 = \alpha.$$

Also write

$$L^{\uparrow}(s, \alpha) = \{(x_0, x_1): -sx_0 + x_1 \geqslant \alpha\}$$

for the positive half-plane determined by $L(s, \alpha)$.

The next two theorems are the main theorems of this section.

THEOREM 1.5. *With notation as in Notation* 1.4, *the following statements hold.*

(1) *The zero dimensional faces in the lower boundary (the extreme points) are precisely the points of the form* $\Psi(p_s^{\pm})$, $s \in \sigma(b)$. *Further, we have*

$$\Psi^{-1}(\Psi(p_s^{\pm})) \cap M_1^+ = \{p_s^{\pm}\}.$$

(2) *The one dimensional faces (the line segments) in the lower boundary are precisely the sets of the form*

$$F = \Psi([p_s^-, p_s^+]), \qquad s \in \sigma_p(b).$$

Further, for each such face we have

$$\Psi^{-1}(F) \cap M_1^+ = [p_s^-, p_s^+]$$

and the slope of F *is* s.

(3) *The map* $(x_0, x_1) \mapsto (1 - x_0, \tau(b) - x_1)$ *takes* B *to itself and interchanges its upper and lower boundaries. This map is a rotation by* π *about the point* $(1/2, \tau(b)/2)$.

(4) *The extreme points on the upper boundary are precisely the points of the form* $\Psi(1 - p_s^{\pm})$, $s \in \sigma(b)$. *Further we have*

$$\Psi^{-1}(1 - p_s^{\pm}) \cap M_1^+ = \{1 - p_s^{\pm}\}.$$

(5) *The line segments on the upper boundary are precisely the sets of the form*

$$F = \Psi[1 - p_s^+, 1 - p_s^-], \qquad s \in \sigma_p(b).$$

Further we have

$$\Psi^{-1}(F) \cap M_1^+ = [1 - p_s^+, 1 - p_s^-]$$

and the slope of F *is* s.

Proof. (1) Fix s in $\sigma(b)$. If $a \in M_1^+$ and $\tau(a) = \tau(p_s^{\pm})$, then $\tau(ba) \geqslant \tau(bp_s^{\pm})$ by part (1) of Lemma 1.3 and therefore $\Psi(p_s^{\pm}) = (\tau(p_s^{\pm}), \tau(bp_s^{\pm}))$ lies on the lower boundary. Furthermore, by part (2) of Lemma 1.3 we have that $\Psi(a) = \Psi(p_s^{\pm})$ only if $a = p_s^{\pm}$. In other words,

$$\Psi^{-1}(\Psi(p_s^{\pm})) \cap M_1^+ = \{p_s^{\pm}\}. \tag{$*$}$$

To see that $\Psi(p_s^{\pm})$ is an extreme point of B, suppose

$$\Psi(p_s^{\pm}) = \lambda \Psi(a_1) + (1 - \lambda)\, \Psi(a_2) = \Psi(\lambda a_1 + (1 - \lambda)\, a_2)$$

for some a_1, a_2 in M_1^+ and $0<\lambda<1$. In this case we have $p_s^\pm=\lambda a_1+(1-\lambda)\,a_2$ by $(*)$ and, since projections are extreme points in M_1^+, we get $a_1=a_2=p_s^\pm$, as desired. To complete the proof of this part of the theorem, we need only show that if $\Psi(a)$ lies on the lower boundary and $\Psi(a)\neq\Psi(p_s^\pm)$ for all $s\in\sigma(b)$, then $\Psi(a)$ is not extreme point. This will be established in the proof of the next part of the theorem, where it will be shown that points on the lower boundary that are not of the form $\Psi(p_s^\pm)$ lie in the interior of line segments.

(2) First, fix $s\in\sigma_p(b)$ so that $p_s^-<p_s^+$, $\Psi(p_s^-)\neq\Psi(p_s^+)$ and $b(p_s^+-p_s^-)=s(p_s^+-p_s^-)$. Observe that if $c\in[p_s^-, p_s^+]$, then $\Psi(c)$ lies on the lower boundary of B by part (1) of Lemma 1.3. Next, write $p_\lambda=\lambda p_s^-+(1-\lambda)\,p_s^+$ for each $0<\lambda<1$. Thus, each p_λ is in $[p_s^-, p_s^+]$ and $\Psi(p_\lambda)$ is a typical point on the line segment joining $\Psi(p_s^-)$ and $\Psi(p_s^+)$. Hence the lower boundary of B contains the line segment joining $\Psi(p_s^-)$ and $\Psi(p_s^+)$. The slope of this line segment is

$$\frac{\tau(bp_s^+)-\tau(bp_s^-)}{\tau(p_s^+)-\tau(p_s^-)}=\frac{\tau(b(b_s^+-p_s^-))}{\tau(p_s^+-p_s^-)}=s,$$

because $b(p_s^+-p_s^-)=s(p_s^+-p_s^-)$.

Let F denote the line segment in the lower boundary of B that contains $\Psi([p_s^-, p_s^+])$ and consider the endpoints of F. We have by Theorem 1.1 that there are projections $p<q$ in M_1^+ such that $[p, q]=\Psi^{-1}(F)\cap M_1^+$ and therefore $p\leqslant p_s^-<p_s^+\leqslant q$. If $p<p_s^-$, then since $p_s^-<p_s^+\leqslant q$ we would have that $\Psi(p_s^-)$ is in the interior of F, which is impossible, because $\Psi(p_s^-)$ is an extreme point by part (1) of this theorem. Hence $p=p_s^-$. Similarly we have $q=p_s^+$. Thus, if $s\in\sigma_p(b)$ then $\Psi([p_s^-, p_s^+])$ is a line segment in the lower boundary of B with slope s. Finally we have, $\Psi^{-1}([p_s^-, p_s^+])\cap M_1^+=[p_s^-, p_s^+]$ by Theorem 1.1.

To complete the proof of this part of the theorem (and finish the proof of part (1)), we must show that line segments of this form together with points of the form $\Psi(p_s^\pm)$ fill out the entire lower boundary of B. To this end, fix a point $(x_0, x_1)=\Psi(a)=(\Psi(a), \Psi(ba))$ on the lower boundary of B such that $(x_0, x_1)\neq\Psi(p_s^\pm)$ for all $s\in\sigma(b)$. In this case we have

$$\tau(p_{s_{\min}}^-)=0<x_0<1=\tau(p_{s_{\max}}^+).$$

Hence, we may write

$$r_1=\sup\{s\in\sigma(b)\colon \tau(p_s^-)\leqslant x_0\}\qquad\text{and}$$

$$r_2=\inf\{s\in\sigma(b)\colon \tau(p_s^+)\geqslant x_0\}.$$

Our next goal is to show that $r_1 = r_2$. Note that since the spectrum is closed, we have $r_i \in \sigma(b)$, $i = 1, 2$. Also, we must have

$$\tau(p_{r_1}^-) \leqslant x_0 \leqslant \tau(p_{r_2}^+)$$

by the definitions of r_1 and r_2. Since $(x_0, x_1) \neq \Psi(p_s^\pm)$ for all $s \in \sigma(b)$ by assumption, we get

$$\tau(p_{r_1}^-) < x_0 < \tau(p_{r_2}^+).$$

Next observe that if $s < r_2$ and $s \in \sigma(b)$, then we have

$$\tau(p_s^+) < x_0 < \tau(p_{r_2}^+) \tag{$*$}$$

by the definition of r_2. Similarly, if $r_1 < s$ and $s \in \sigma(b)$, then

$$\tau(p_{r_1}^-) < x_0 < \tau(p_s^-) \tag{$**$}$$

by the definition of r_1. Now suppose that $r_1 < r_2$. In this case since each r_i is in $\sigma(b)$, we would get

$$\tau(p_{r_1}^+) < x_0 < \tau(p_{r_2}^-) \tag{$***$}$$

by putting $s = r_1$ in ($*$) and $s = r_2$ in ($**$). On the other hand, if we had $r_1 < s < r_2$ and $s \in \sigma(b)$, then using ($*$) and ($**$) we would get

$$\tau(p_s^+) < x_0 < \tau(p_s^-).$$

Since $p_s^- \leqslant p_s^+$, this is impossible and so no such s exists. Thus if $r_1 < r_2$, then we must have that r_1 and r_2 form the endpoints of a gap in the spectrum of b. But in this case we would have $p_{r_1}^+ = p_{r_2}^-$ and

$$\tau(p_{r_1}^+) = \tau(p_{r_2}^-),$$

contradicting ($***$). Hence we must have $r_1 = r_2 = r$.

With this we have

$$\tau(p_r^-) < x_0 < \tau(p_r^+)$$

and therefore $p_r^- < p_r^+$. Hence r is an eigenvalue for b. Therefore $\Psi([p_r^-, p_r^+])$ is a line segment in the lower boundary by the first part of the proof of this part of the theorem. Since $\tau(p_r^-) < x_0 < \tau(p_r^+)$ and (x_0, x_1) lies on the lower boundary, we get that (x_0, x_1) lines in interior of this line segment, as desired.

(3) For any $a \in M_1^+$, $\Psi(a) = (\tau(a), \tau(ba))$ is in B and $\Psi(1-a) = (1-\tau(a), \tau(b)-\tau(ba))$ is in B. Thus, the map $(x_0, x_1) \mapsto (1-x_0, \tau(b)-x_1)$ takes $\Psi(a)$ to $\Psi(1-a)$ and hence takes B onto itself. It is easy to check that this map is a rotation by π about the point $(1/2, \tau(b)/2)$ and with this it is evident that it switches upper and lower boundaries.

The final assertions in parts (4) and (5) follow immediately from parts (1) and (2) and the symmetry property of B described in part (3). ∎

Remark. Observe that if r is a real number such that r is not in the spectrum of b, then $p_r^- = p_r^+ = p_r$ and there is an $s \in \sigma(b)$ such that $p_r = p_s^-$ or $p_r = p_s^+$. Indeed, since $r \notin \sigma(b)$, it is clear that $p_r^- = p_r^+$. If $r < s_{\min}$, then $p_r = 0 = p_{s_{\min}}^-$. Similarly, if $r > s_{\max}$, then $p_r = 1 = p_{s_{\max}}^+$. If $s_{\min} < r < s_{\max}$, then r must lie in a gap in the spectrum of the form (s_1, s_2) as described below. In this case we have $p_r = p_{s_1}^+ = p_{s_2}^-$. Combining this with part (1) of Theorem 1.5, we get that $\Psi(p_s^\pm)$ is an extreme point on the lower boundary of B for all real s.

In the next theorem we shall discuss the geometry of the lower boundary curve in terms of its differentiability. As usual a *corner* on the boundary of a compact convex subset of $\mathbb{R}^2$ is a point that admits two distinct lines of support and a *gap* in the spectrum of b is a real interval of the form (s_1, s_2) such that each s_i is in $\sigma(b)$, but $(s_1, s_2) \cap \sigma(b) = \varnothing$.

THEOREM 1.6. *With notation as in Notation* 1.4, *the following statements hold.*

(1) *If $s > s_{\min}$, then the left derivative of the lower boundary function f at $\tau(p_s^-)$ exists and is given by the formula*

$$f'_-(\tau(p_s^-)) = \sup\{s' \in \sigma(b) : s' < s\}.$$

If $s < s_{\max}$, then the right derivative of the lower boundary function f at $\tau(p_s^+)$ exists and is given by the formula

$$f'_+(\tau(p_s^+)) = \inf\{s' \in \sigma(b) : s' > s\}.$$

(2) *The points of nondifferentiability (corners) on the lower boundary curve are in one-to-one correspondence with the gaps in the spectrum of b.*

(3) *For each real s, $\tau((b-s1)\,p_s^-) = \tau((b-s1)\,p_s^+)$. The line $L(s, \alpha)$ is a line of support for B such that $B \subset L^\uparrow(s, \alpha)$ if and only if*

$$\alpha = \tau((b-s1)\,p_s^\pm).$$

In this case $L(s, \alpha)$ passes through $\Psi(p_s^{\pm})$. Moreover, we have

$$\Psi^{-1}(L(s, \alpha)) \cap M_1^+ = [p_s^-, p_s^+]$$

and $\Psi([p_s^-, p_s^+]) = L(s, \alpha) \cap B$.

(4) *The set B has corners at $\Psi(0)$ and $\Psi(1)$. The lines $L(s_{\min}, 0)$ and $L(s_{\max}, 0)$ are lines of support for B at $\Psi(0)$. The lines $L(s_{\max}, \tau(b - s_{\max} 1))$ and $L(s_{\min}, \tau(b - s_{\min} 1))$ are lines of support for B at $\Psi(1)$.*

Proof. (1) Since B is convex, the lower boundary function f is a convex function and therefore for each $x \in (0, 1)$,

$$\frac{f(x) - f(x-h)}{h}$$

increases as h decreases to zero. Furthermore, the convexity of f also implies that as h varies this quantity is bounded above by

$$\frac{f(1) - f(x)}{1 - x}.$$

Since bounded monotone sequences converge, this shows that left sided derivatives exist at every point, including 1. A similar argument shows that right sided derivatives also exist at every point, including 0.

Fix $s \in \mathbb{R}$ such that $s_{\min} < s$. In this case $\sigma(b) \cap (-\infty, s) \neq \emptyset$ and we may define

$$r = \sup \{s' \in \sigma(b) \colon s' < s\}.$$

Note that r is in $\sigma(b)$ because the spectrum is closed. First suppose $(r - \varepsilon, r) \cap \sigma(b) = \emptyset$ for some $\varepsilon > 0$. If we had $r = s$, then we would get $(r - \varepsilon, r) \cap \sigma(b) = (s - \varepsilon, s) \cap \sigma(b) = \emptyset$ contradicting the definition of r. Hence we have $r < s$ in this case. Since the definition of r also implies that $(r, s) \cap \sigma(b) = \emptyset$, we get that r is an isolated point in $\sigma(b)$. Hence $r \in \sigma_p(b)$, $p_r^+ = p_s^-$, and $\Psi(p_s^-) = \Psi(p_s^+)$ is the right hand endpoint of a line segment in the lower boundary with slope r by part (2) of Theorem 1.5. Thus the left derivative of f at $\tau(p_s^-)$ is r, in this case.

Otherwise we can choose a sequence of distinct points $\{r_n\} \subset \sigma(b)$ increasing to r. The points $\Psi(p_{r_n}^-)$ lie on the lower boundary by part (1) of Theorem 1.5. Moreover, since $p_{r_n}^-$ converges to p_r^- in the weak*-topology, and τ is normal, $\alpha_n = \tau(p_{r_n}^-)$ converges to $\alpha = \tau(p_r^-)$. However, α_n does not equal α for any n because τ is faithful. Since $p_{r_n}^-$ and p_r^- are spectral projections of b, we have

$$r_n(p_r^- - p_{r_n}^-) \leqslant b(p_r^- - p_{r_n}^-) \leqslant r(p_r^- - p_{r_n}^-)$$

for each n. Applying τ yields

$$r_n(\alpha - \alpha_n) \leqslant f(\alpha) - f(\alpha_n) \leqslant r(\alpha - \alpha_n)$$

and so

$$r_n \leqslant \frac{f(\alpha) - f(\alpha_n)}{\alpha - \alpha_n} \leqslant r.$$

Since $r_n \to r$ and $\alpha_n \to \alpha$ we conclude that the left derivative at $\alpha = \tau(p_s^-)$ equals r.

A similar argument proves the analogous statement for right derivatives.

(2) This follows immediately from part (1) of this theorem.

(3) Fix a real number s. If $s \in \sigma_p(b)$, then we have $b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-)$. Further, if $s \notin \sigma_p(b)$, then we have $p_s^- = p_s^+$ and so we again get $b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-)$. Hence, in all cases we have $(b - s1)(p_s^+ - p_s^-) = 0$ and so $\tau((b - s1)\, p_s^+) = \tau((b - s1)\, p_s^-)$. Next, if $\alpha = \tau((b - s1)\, p_s^\pm)$, then we have:

$$-s\tau(p_s^\pm) + \tau(bp_s^\pm) = \tau((b - s1)\, p_s^\pm) = \alpha$$

and so $\Psi(p_s^\pm)$ lies on $L(s, \alpha)$. Our next goal is to show that $L(s, \alpha)$ is a line of support for B whose upper half-space contains B. To see this, it is useful to consider several cases.

First suppose $s \in \sigma_p(b)$. In this case $\Psi(p_s^-)$ and $\Psi(p_s^+)$ are the endpoints of a line segment in the lower boundary whose slope is s by part (2) of Theorem 1.5. Since $L(s, \alpha)$ passes through $\Psi(p_s^\pm)$ and has slope s, $L(s, \alpha)$ contains this line segment and is tangent to the lower boundary of B. As this line supports B on the lower boundary, $B \subset L^\uparrow(s, \alpha)$.

Observe that if s is an isolated point in $\sigma(b)$, then $s \in \sigma_p(b)$. Hence, the desired conclusions hold in this case by the previous paragraph.

Next suppose $s \in \sigma(b) \backslash \sigma_p(b)$ so that s is not an isolated point in $\sigma(b)$. Since $s \notin \sigma_p(b)$ we have $p_s^- = p_s^+ = p_s$. Since s is not isolated in $\sigma(b)$, we may use part (1) of this theorem to conclude that at least one of the one sided derivatives of the lower boundary function takes the value s at $\tau(p_s)$. Hence B admits a line of support at $\Psi(p_s)$ with slope s. Arguing as above, we get that this line is $L(s, \alpha)$ and $B \subset L^\uparrow(s, \alpha)$. Thus, the desired conclusions hold for all $s \in \sigma(b)$.

Next suppose that (s_1, s_2) is a gap in the spectrum of b. In this case we have $p_{s_1}^+ = p_{s_2}^-$. Further, if $\alpha_1 = \tau((b - s_1)\, p_{s_1}^+)$ and $\alpha_2 = \tau((b - s_2)\, p_{s_2}^-)$, then we have that $L(s_1, \alpha_1)$ and $L(s_2, \alpha_2)$ are lines of support passing through $\Psi(p_{s_1}^+) = \Psi(p_{s_2}^-)$ because s_1 and s_2 are in the spectrum of b. Hence if L is a line passing through $\Psi(p_{s_1}^+)$ whose slope lies between s_1 and s_2, then L is a line of support for B that contains B in its upper half-space. Since $L(s, \alpha)$ is such a line for all $s_1 < s < s_2$, the desired conclusions hold for all s such that $s_1 < s < s_2$.

As $s_{\min} \in \sigma(b)$, we have $p_s^- = 0$, $L(s_{\min}, 0)$ is a line of support for B at $\Psi(0)$ and B lies in $L^{\uparrow}(s_{\min}, 0)$. Now suppose $s < s_{\min}$. In this case we have $p_s^- = p_s^+ = 0$ so that $L(s, 0)$ passes through $\Psi(0)$. Further, since $L(s_{\min}, 0)$ supports B on its lower boundary and $s < s_{\min}$, $L(s, 0)$ is a line of support for B and $B \subset L^{\uparrow}(s, 0)$. Similarly, $L(s_{\max}, \tau(b - s_{\max} 1))$ is a line of support for B at $\Psi(1)$ and if $s > s_{\max}$, then $p_s^- = p_s^+ = 1$ and $L(s, \tau(b - s1))$ is also a line of support for B whose upper half-space contains B.

Thus, in each case we get that if $\alpha = \tau((b - s1)\, p_s^{\pm})$, then $L(s, \alpha)$ is a line of support for B and $B \subset L^{\uparrow}(s, \alpha)$. Since all possible cases have been checked, the proof of the "if" portion of the second assertion in part (3) is complete.

Conversely, observe that with s fixed, as β ranges over $\mathbb{R}$, the lines $L(s, \beta)$ are all parallel. Thus, there is exactly one β for which $L(s, \beta)$ is a line of support for B and $B \subset L^{\uparrow}(s, \beta)$. Since the line $L(s, \alpha)$ has these properties by the preceding paragraphs, we must have $\beta = \alpha$. This completes the proof of the first three assertions in part (3).

For the final assertion, observe that the (possibly degenerate) line segment $F = L(s, \alpha) \cap B$ is a face in B. Hence, F is either an extreme point or else it is a line segment. If F is an extreme point, then it must have the form $\{\Psi(p_s^{\pm})\}$ for some $s \in \sigma(b)$ by part (1) of Theorem 1.5. Moreover, we must have $p_s^- = p_s^+$, since otherwise F would be a line segment by part (2) of Theorem 1.5. Hence, in this case we have $F = \{\Psi(p_s^-)\} = \{\Psi(p_s^+)\}$ and

$$\Psi^{-1}(F) \cap M_1^+ = \Psi^{-1}(L(s, \alpha)) \cap M_1^+ = \{p_s^{\pm}\} = [p_s^-, p_s^+]$$

by part (1) of Theorem 1.5. Similarly, if F is a line segment, then we have $F = \Psi([p_s^-, p_s^+])$ for some s in $\sigma_p(b)$ and

$$\Psi^{-1}(F) \cap M_1^+ = \Psi^{-1}(L(s, \alpha)) \cap M_1^+ = [p_s^-, p_s^+]$$

by part (2) of Theorem 1.5.

(4) We have that $L(s_{\min}, 0)$ and $L(s_{\max}, \tau(b - s_{\max} 1))$ are lines of support for B at $\Psi(0)$ and $\Psi(1)$, respectively, by part (3) of this theorem. Next, using the symmetry of B described in part (3) of Theorem 1.5, we get that $L(s_{\max}, 0)$ is a line of support for B at $\Psi(0)$ and $L(s_{\min}, \tau(b - s_{\min} 1))$ is a line of support for B at $\Psi(1)$. If $s_{\min} = s_{\max} = s$, then $b = s1$ and B is the line segment joining $(0, 0)$ and $(1, \tau(b) = (1, s))$. In this case $\Psi(0)$ and $\Psi(1)$ are the endpoints of this line segment and so are degenerate corners of B.

Otherwise we have $s_{\min} < s_{\max}$, and again, there are two distinct lines of support for B at $\Psi(0)$ and $\Psi(1)$. ∎

We now summarize how the geometry of B and spectral data for b are related. There are five mutually exclusive possibilities. In each case a geometric property of the lower boundary corresponds to an equivalent spectral property of the operator b. We have already reviewed the notations of a corner on the lower boundary of B and a gap in $\sigma(b)$. A *flat spot* on the lower boundary is a nonempty line segment on the lower boundary curve and a *round point* is a point that does not lie in the interior of a flat spot and where the lower boundary curve is differentiable.

THEOREM 1.7. *As above, let f denote the lower boundary function associated to the set B. If x is in $(0, 1)$, then the following assertions hold.*

(1) *The point $(x, f(x))$ is in the interior of a flat spot with slope s if and only if f is differentiable at x, there exists a unique $s \in \sigma_p(b)$ such that $\tau(p_s^-) < x < \tau(p_s^+)$ and $f'(x) = s$.*

(2) *The point $(x, f(x))$ is a round spot at the left end of a flat spot with slope s if and only if f is differentiable at x, there exists unique $s \in \sigma_p(b)$ such that $\tau(p_s^-) = x < \tau(p_s^+)$ and $f'(x) = s$.*

(3) *The point $(x, f(x))$ is a round spot at the right end point of a flat spot with slope s if and only if f is differentiable at x, there exists unique $s \in \sigma_p(b)$ such that $\tau(p_s^-) < x = \tau(p_s^+)$ and $f'(x) = s$.*

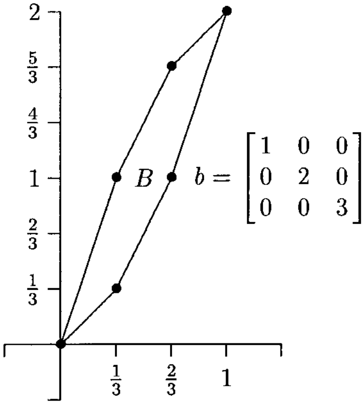
(4) *The point $(x, f(x))$ is a round spot that is not the endpoint of a flat spot and the tangent line at $(x, f(x))$ has slope s if and only if f is differentiable at x, there exists a unique s in $\sigma(b) \backslash \sigma_p(b)$ such that $\tau(p_s^-) = x = \tau(p_s^+)$ and $f'(x) = s$.*

(5) *The point $(x, f(x))$ is a corner such that the slopes of the lines of support for B at $(x, f(x))$ lie between s_1 and s_2 if and only if f is not differentiable at x, there exists unique gap in $\sigma(b)$ with endpoints $s_1 < s_2$, $\tau(p_{s_1}^+) = x = \tau(p_{s_2}^-)$, the left derivative of f at x is s_1 and the right derivative of f at x is s_2.*

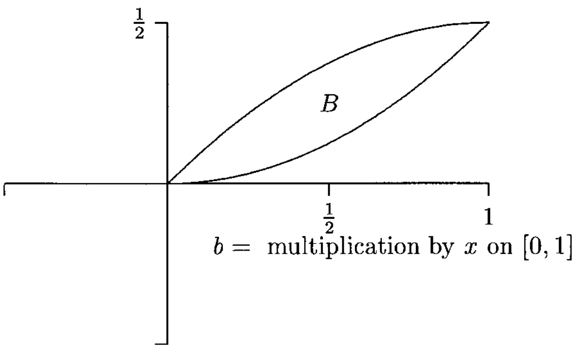
Proof. The statements as follow easily from Theorems 1.5 and 1.6. ∎

Remark. Write $N = \{b\}''$, the von Neumann algebra generated by b and the identity. Since the extreme points of B are images of spectral projections of b, and Ψ is weakly continuous, we get that $B = \Psi(M_1^+) = \Psi(N_1^+)$.

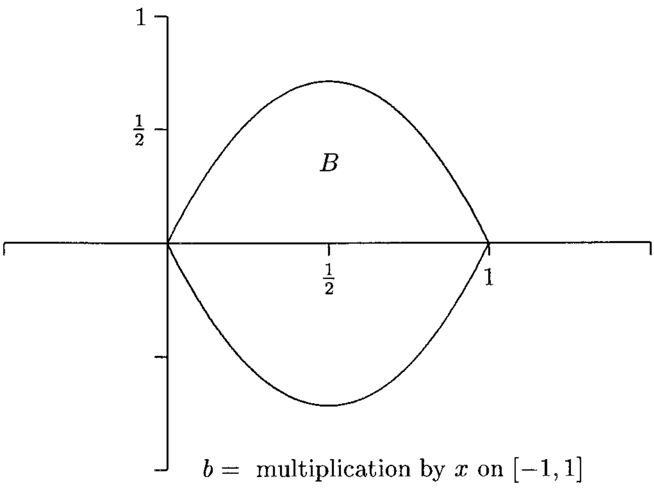
We conclude this section by presenting several illustrations of the set B.



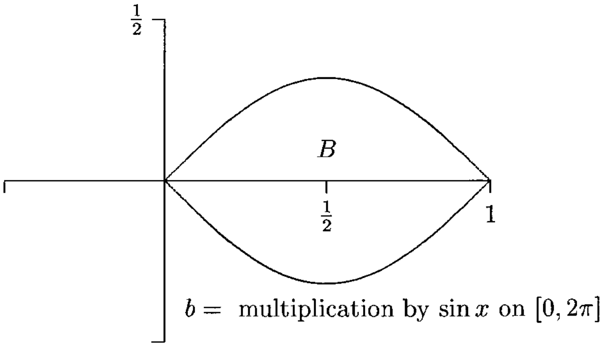
$b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
B
2
$\frac{5}{3}$
$\frac{4}{3}$
1
$\frac{2}{3}$
$\frac{1}{3}$
$\frac{1}{3}$ $\frac{2}{3}$ 1



$\frac{1}{2}$
B
$\frac{1}{2}$ 1
$b =$ multiplication by x on $[0, 1]$



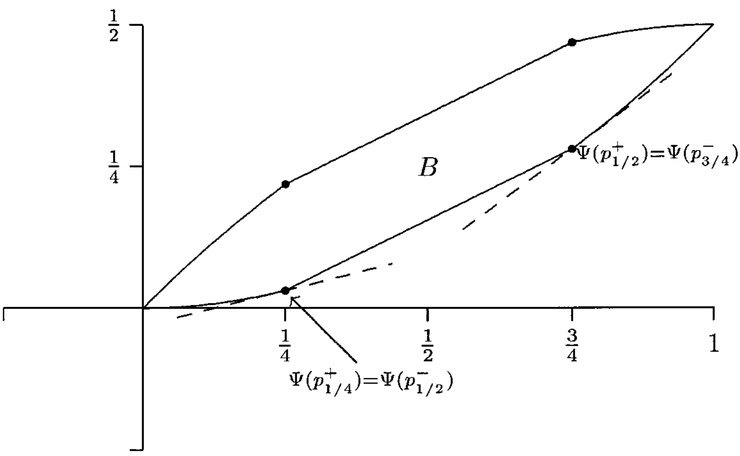
1
$\frac{1}{2}$
B
$\frac{1}{2}$ 1
$b =$ multiplication by x on $[-1, 1]$



In the next example we consider the measure space $\{X, \mu\}$, where

$$X = [0, 1/4] \cup \{1/2\} \cup [3/4, 1]$$

μ is Lebesgue measure on $[0, 1/4] \cup [3/4, 1]$ and $\mu\{1/2\} = 1/2$.



2. GENERALIZATIONS TO HIGHER DIMENSIONS: THE SPECTRAL SCALE

We now turn to the case when $n \geqslant 1$. Let us begin with a formal definition of the object of interest.

DEFINITION 2.1. We continue to use M to denote a finite von Neumann algebra with faithful normal tracial state τ. Fix self-adjoint elements $b_1, ..., b_n$ in M. The *spectral scale* of $b_1, ..., b_n$ *relative to* τ is the set $B = \Psi(M_1^+) \subset \mathbb{R}^{n+1}$ where Ψ: $M \to \mathbb{C}^{n+1}$ is defined by

$$\Psi(a) = (\tau(a), \tau(b_1 a), ..., \tau(b_n a)).$$

Also, for each nonzero vector $\mathbf{t} = (t_1, ..., t_n) \in \mathbb{R}^n$ write

$$b_{\mathbf{t}} = t_1 b_1 + \cdots + t_n b_n.$$

Next, define the map $\Psi_{\mathbf{t}}$ from M to $\mathbb{C}^2$ by $\Psi_{\mathbf{t}}(a) = (\tau(a), \tau(b_{\mathbf{t}} a))$ and write $B_{\mathbf{t}} = \Psi_{\mathbf{t}}(M_1^+)$. Finally, write $\pi_{\mathbf{t}}$ for the map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^2$ by

$$\pi_{\mathbf{t}}(x_0, x_1, ..., x_n) = (x_0, t_1 x_1 + \cdots + t_n x_n).$$

The basic tool for studying the spectral scale in $n+1$ dimensions is a reduction to the two-dimensional case considered in Section 1, effected by means of the preceding definitions. The following lemma is basic.

LEMMA 2.2. *We have* $\Psi_{\mathbf{t}} = \pi_{\mathbf{t}} \circ \Psi$ *and* $B_{\mathbf{t}} = \pi_{\mathbf{t}}(B)$.

Proof. For the first assertion we have

$$\begin{aligned}\Psi_{\mathbf{t}}(a) &= (\tau(a), \tau(b_{\mathbf{t}} a)) \\ &= (\tau(a), \tau((t_1 b_1 + \cdots + t_n b_n)\, a)) \\ &= (\tau(a), t_1 \tau(b_1 a) + \cdots + t_n \tau(b_n a)) \\ &= \pi_{\mathbf{t}}(\tau(a), \tau(b_1 a), ..., \tau(b_n a)) \\ &= \pi_{\mathbf{t}}(\Psi(a)).\end{aligned}$$

Since

$$B_{\mathbf{t}} = \Psi_{\mathbf{t}}(M_1^+) = \pi_{\mathbf{t}}(\Psi(M_1^+)) = \pi_{\mathbf{t}}(B),$$

the second assertion is true. ∎

For any nonzero $\mathbf{t} \in \mathbb{R}^n$ and any $s \in \mathbb{R}$ let $p_{\mathbf{t},s}^+$ and $p_{\mathbf{t},s}^-$ denote the spectral projections of $b_{\mathbf{t}}$ determined by the intervals $(-\infty, s]$ and $(-\infty, s)$. Also write $P(\mathbf{t}, s, \alpha)$ for the hyperplane in $\mathbb{R}^{n+1}$ defined by the formula

$$-s x_0 + t_1 x_1 + \cdots + t_n x_n = \alpha$$

and let $P^{\uparrow}(\mathbf{t}, s, \alpha)$ denote the half-space defined by the inequality

$$-sx_0 + t_1x_1 + \cdots + t_nx_n \geqslant \alpha.$$

The following is a generalization of portions of Theorems 1.5 and 1.6.

THEOREM 2.3. *If M is a finite von Neumann algebra with faithful normal tracial state τ, $b_1, ..., b_n$ are self-adjoint elements of M and B denotes their spectral scale relative to τ, then the following statements hold for each non-zero vector $\mathbf{t} \in \mathbb{R}^n$ and each real s.*

(1) *If $\mathbf{x}$ is an extreme point of B, then there is a projection p in M such that*

$$\Psi(p) = \mathbf{x} \qquad \text{and} \qquad \Psi^{-1}(\mathbf{x}) \cap M_1^+ = \{p\}.$$

Further, $\Psi(p^+_{\mathbf{t},s})$ and $\Psi(p^-_{\mathbf{t},s})$ are extreme points of B.

(2) *We have $\tau((b_{\mathbf{t}} - s1)\, p^+_{\mathbf{t},s}) = \tau((b_{\mathbf{t}} - s1)\, p^-_{\mathbf{t},s})$. The hyperplane $P(\mathbf{t}, s, \alpha)$ is a hyperplane of support for B with $B \subset P^{\uparrow}(\mathbf{t}, s, \alpha)$ if and only if*

$$\alpha = \tau((b_{\mathbf{t}} - s1)\, p^{\pm}_{\mathbf{t},s}).$$

In this case we have $\Psi(p^{\pm}_{\mathbf{t},s}) \in P(\mathbf{t}, s, \alpha)$.

(3) *If $\alpha = \tau((b_{\mathbf{t}} - s1)\, p^{\pm}_{\mathbf{t},s})$, then $F = P(\mathbf{t}, s, \alpha) \cap B$ is a face in B. Moreover,*

$$\Psi^{-1}(P(\mathbf{t}, s, \alpha)) \cap M_1^+ = [p^-_{\mathbf{t},s}, p^+_{\mathbf{t},s}]$$

and

$$F = \Psi([p^-_{\mathbf{t},s}, p^+_{\mathbf{t},s}]).$$

(4) *If we put $\beta = \tau((b_{\mathbf{t}} - s1)(1 - p^{\pm}_{\mathbf{t},s}))$, then the hyperplane $P(\mathbf{t}, s, \beta)$ is a hyperplane of support for B that passes through the points $\Psi(1 - p^{\pm}_{\mathbf{t},s})$. This hyperplane is parallel to the supporting hyperplane $P(\mathbf{t}, s, \alpha)$ and we have $B \subset P^{\downarrow}(\mathbf{t}, s, \beta)$.*

Proof. (1) Fix an extreme point $\mathbf{x}$ in B. Since $\{\mathbf{x}\}$ is a face of B there are projections $p \leqslant q$ in M such that

$$\Psi^{-1}(\mathbf{x}) \cap M_1^+ = [p, q]$$

by Theorem 1.1. Since $\Psi(p) = \mathbf{x} = \Psi(q)$, we get that $\tau(p) = \tau(q)$. Further, since τ is faithful and $p \leqslant q$, we must have $p = q$. Hence

$$\Psi^{-1}(\mathbf{x}) \cap M_1^+ = \{p\},$$

as desired.

For the final assertion in part (1) suppose $\Psi(p^{\pm}_{\mathbf{t},s}) = \Psi(\lambda a_1 + (1-\lambda)\,a_2)$, where $0 < \lambda < 1$ and the a_i's are in M_1^+. In this case, we have

$$\Psi_{\mathbf{t}}(p^{\pm}_{\mathbf{t},s}) = \pi_{\mathbf{t}}(\Psi(p^{\pm}_{\mathbf{t},s})) = \pi_{\mathbf{t}}(\Psi(\lambda a_1 + (1-\lambda)\,a_2)) = \Psi_{\mathbf{t}}(\lambda a_1 + (1-\lambda)\,a_2)$$

and therefore $p^{\pm}_{\mathbf{t},s} = \lambda a_1 + (1-\lambda)\,a_2$ by part (2) of Lemma 1.3. As projections are extreme points in M_1^+, we get $p^{\pm}_{\mathbf{t},s} = a_1 = a_2$, as desired.

(2) We have $(b_{\mathbf{t}} - s1)(p^{+}_{\mathbf{t},s} - p^{-}_{\mathbf{t},s}) = 0$, so that $\tau((b_{\mathbf{t}} - s1)\,p^{+}_{\mathbf{t},s}) = \tau((b_{\mathbf{t}} - s1)\,p^{-}_{\mathbf{t},s})$. Now suppose α denotes this common value. With this we get that

$$-s\tau(p^{\pm}_{\mathbf{t},s}) + t_1\tau(b_1 p^{\pm}_{\mathbf{t},s}) + \cdots + t_n\tau(b_n p^{\pm}_{\mathbf{t},s}) = -s\tau(p^{\pm}_{\mathbf{t},s}) + \tau(b_{\mathbf{t}} p^{\pm}_{\mathbf{t},s}) = \alpha$$

and therefore $\Psi(p^{\pm}_{\mathbf{t},s}) \in P(\mathbf{t}, s, \alpha)$. Let $L_{\mathbf{t}}(s, \alpha)$ denote the line in $\mathbb{R}^2$ determined by the equation

$$-sx_0 + x_1 = \alpha$$

and note that by Part (2) of Theorem 1.6, $L_{\mathbf{t}}(s, \alpha)$ is a line of support for $B_{\mathbf{t}}$ and that $B_{\mathbf{t}}$ lies in $L_{\mathbf{t}}^{\uparrow}(s, \alpha)$. Now fix $a \in M_1^+$. Since $\Psi_{\mathbf{t}}(a)$ lies in this upper half-plane, we have

$$\alpha \leqslant -s\tau(a) + \tau(t_1 b_1 a + \cdots + t_n b_n a) = -s\tau(a) + t_1\tau(b_1 a) + \cdots + t_n\tau(b_n a)$$

and so $\Psi(a) \in P^{\uparrow}(\mathbf{t}, s, \alpha)$. Hence, $P(\mathbf{t}, s, \alpha)$ has the desired properties.

Conversely, observe that with $\mathbf{t}$ and s fixed, as β ranges over $\mathbb{R}$, the hyperplanes $P(\mathbf{t}, s, \beta)$ are all parallel. Thus, there is exactly one β for which $P(\mathbf{t}, s, \beta)$ is a hyperplane of support for B and $B \subset P^{\uparrow}(\mathbf{t}, s, \beta)$. Since the hyperplane $P(\mathbf{t}, s, \alpha)$ has these properties by the preceding paragraph, we must have $\beta = \alpha$.

(3) Since $P(\mathbf{t}, s, \alpha)$ is a supporting hyperplane for B by part (2), $F = P(\mathbf{t}, s, \alpha) \cap B$ is a face of B. Hence, there are unique projections $p \leqslant q$ in M such that

$$\Psi^{-1}(F) \cap M_1^+ = [p, q]$$

and $\Psi([p, q]) = F$ by Theorem 1.1. If $a \in M_1^+$ and $\Psi(a) \in P(\mathbf{t}, s, \alpha)$, then $\Psi(a) \in P(\mathbf{t}, s, \alpha) \cap B$ and therefore

$$\begin{aligned}[p, q] &= \Psi^{-1}(F) \cap M_1^+ = \Psi^{-1}(P(\mathbf{t}, s, \alpha) \cap B) \cap M_1^+ \\ &= \Psi^{-1}(P(\mathbf{t}, s, \alpha)) \cap M_1^+.\end{aligned}$$

To complete the proof of this part, we need only show that $p = p^-_{\mathbf{t},s}$ and $q = p^+_{\mathbf{t},s}$. For this, note that since $\Psi(p^\pm_{\mathbf{t},s}) \in F$ by part (2), we get

$$p \leqslant p^-_{\mathbf{t},s} \leqslant p^+_{\mathbf{t},s} \leqslant q.$$

Next observe that we have

$$L_{\mathbf{t}}(s, \alpha) \cap B_{\mathbf{t}} = \Psi_{\mathbf{t}}([p^-_{\mathbf{t},s}, p^+_{\mathbf{t},s}])$$

by part (2) of Theorem 1.6. On the other hand, we have $\pi_{\mathbf{t}}(P(\mathbf{t}, s, \alpha)) = L_{\mathbf{t}}(s, \alpha)$ and $\pi_{\mathbf{t}}(B) = B_{\mathbf{t}}$ and so

$$\begin{aligned}\Psi_{\mathbf{t}}([p, q]) &= \pi_{\mathbf{t}}(\Psi([p, q])) = \pi_{\mathbf{t}}(F)\\ &= \pi_{\mathbf{t}}(\Psi(P(\mathbf{t}, s, \alpha) \cap B)) \subset L_{\mathbf{t}}(s, \alpha) \cap B_{\mathbf{t}}\\ &= \Psi_{\mathbf{t}}([p^-_{\mathbf{t},s}, p^+_{\mathbf{t},s}]),\end{aligned}$$

and therefore $p^-_{\mathbf{t},s} \leqslant p \leqslant q \leqslant p^+_{\mathbf{t},s}$. Hence $p^-_{\mathbf{t},s} = p$ and $p^+_{\mathbf{t},s} = q$.

(4) If $\gamma = \tau((b_{-\mathbf{t}} - (-s)\,1)\, p^\pm_{-\mathbf{t},-s})$, then we have that $P(-\mathbf{t}, -s, \gamma)$ is a hyperplane of support for B that contains $\Psi(p^\pm_{-\mathbf{t},-s})$ and $B \subset P^\uparrow(-\mathbf{t}, -s, \gamma)$ by part (2) of this theorem. Now observe that $b_{-\mathbf{t}} = -b_{\mathbf{t}}$ and $p^\pm_{-\mathbf{t},-s} = 1 - p^\mp_{\mathbf{t},s}$. Hence $\gamma = \tau((b_{-\mathbf{t}} - (-s)\,1)\, p^\pm_{-\mathbf{t},-s}) = -\tau((b_{\mathbf{t}} - s1)(1 - p^\pm_{\mathbf{t},s})) = -\beta$ and so $P(-\mathbf{t}, -s, \gamma) = P(-\mathbf{t}, -s, -\beta) = P(\mathbf{t}, s, \beta)$ and $P^\uparrow(-\mathbf{t}, -s, -\beta) = P^\downarrow(\mathbf{t}, s, \beta)$. ▮

THEOREM 2.4. *If M, τ, and $b_1, ..., b_n$ are as in Theorem* 2.3, *and N is the von Neumann subalgebra of M generated by $b_1, ..., b_n$ and the identity, then* $\Psi(M_1^+) = \Psi(N_1^+)$.

Proof. This follows easily from Theorem 3.2 below. It may also be established as follows. Since M is finite, τ is normal and $N \subset M$, there is a conditional expectation E that maps M onto N and preserves the trace [Tak, Proposition 2.36, page 332]. Specifically, we have

$$\tau(a) = \tau(E(a))$$

for each a in M. The map E also has the property that $E(ba) = bE(a)$ for $a \in M$ and $b \in N$ [Tak, Theorem 3.4, p. 131]. Hence

$$\begin{aligned}\Psi(a) &= (\tau(a), \tau(b_1 a), ..., \tau(b_n a))\\ &= (\tau(E(a)), \tau(E(b_1 a)), ..., \tau(E(b_n a)))\\ &= (\tau(E(a)), \tau(b_1 E(a)), ..., \tau(b_n E(a)))\\ &= \Psi(E(a)). \quad ▮\end{aligned}$$

3. INVARIANCE PROPERTIES OF THE SPECTRAL SCALE

A natural and fundamental question to ask at this point is: Does the spectral scale "determine" the n-tuple $b_1, ..., b_n$? Our purpose in this section is to address this question. It is convenient to change our notation slightly. Let us begin by formally recording some of the new notation that will be employed.

Notation 3.1. (1) Throughout this section M and N continue to stand for finite von Neumann algebras with finite faithful normal tracial states τ_M and τ_N. However, we no longer assume that $N \subset M$. Instead, we shall assume that

$$M = \{b_1, ..., b_n\}'' \qquad \text{and} \qquad N = \{c_1, ..., c_n\}'',$$

where $b_1, ..., b_n$ (resp., $c_1, ..., c_n$) are self-adjoint.

(2) We write B and C for the spectral scales of $b_1, ..., b_n$ and $c_1, ..., c_n$ relative to τ_M and τ_N determined by the maps Ψ_M and Ψ_N as defined in Definition 2.1.

(3) Recall that the states τ_M and τ_N determine representations of M and N via the GNS construction. We write $\{\pi_M, H_M, \xi_M\}$ and $\{\pi_N, H_N, \xi_N\}$ for the representations, Hilbert spaces, and canonical cyclic vectors that arise from this construction. Thus, for $a \in M$, we have

$$\pi_M(a) \in B(H_M) \qquad \text{and} \qquad \tau_M(a) = (\pi_M(a)\, \xi_M, \xi_M).$$

(4) Finally, recall that if π_1 and π_2 are representations of a C^*-algebra A on Hilbert spaces H_1 and H_2, then π_1 and π_2 are *equivalent* if there is a unitary transformation u mapping H_1 to H_2 such that

$$u\pi_1(x) = \pi_2(x)\, u$$

for each $x \in A$. Let us abuse this notation slightly and say that the tracial representations of M and N are *equivalent* if there is a unitary transformation mapping H_M onto H_N and such that

$$u\xi_M = \xi_N \qquad \text{and} \qquad u\pi_M(b_i) = \pi_N(c_i)\, u, \qquad i = 1, ..., n.$$

Thus, in this section we shall assume that the spectral scales, B and C are equal and ask if the tracial representations are equivalent. It turns out that the answer is yes if M and N are abelian, but a simple 3×3 matrix example shows that the answer in the general case is no. However, if we pass to matrices over M and N and require that the spectral scales be "completely equal," then it will be shown in Theorem 3.11 that the tracial representations are equivalent even in the noncommutative case.

We begin our study by presenting some equivalent reformulations of the assumption that $B = C$, i.e., different ways of viewing the information content of the spectral scale. As in the previous section we write $p^{\pm}_{\mathbf{t},s}$ for the spectral projection of $b_{\mathbf{t}}$ corresponding to the interval $(-\infty, s]$ or $(-\infty, s)$. Let us write $q^{\pm}_{\mathbf{t}}$ for the analogous spectral projection of $c_{\mathbf{t}}$.

THEOREM 3.2. *With notation as in Notation* 3.1 *and Section* 2, *the following statements are equivalent.*

(1) $B = C$;

(2) $B_{\mathbf{t}} = C_{\mathbf{t}}$ *for each nonzero* $\mathbf{t}$ *in* $\mathbb{R}^n$;

(3) $\tau_M(p^{\pm}_{\mathbf{t},s}) = \tau_N(q^{\pm}_{\mathbf{t},s})$ *for each* s *in* $\mathbb{R}$ *and each nonzero* $\mathbf{t}$ *in* $\mathbb{R}^n$;

(4) $\tau_M(b^k_{\mathbf{t}}) = \tau_N(c^k_{\mathbf{t}})$ *for each* k *in* $\mathbb{N}$ *and each nonzero* $\mathbf{t}$ *in* $\mathbb{R}^n$;

(5) $\tau_M(f(b_{\mathbf{t}})) = \tau_N(f(c_{\mathbf{t}}))$ *for each nonzero* $\mathbf{t}$ *in* $\mathbb{R}^n$ *and each bounded Borel function* f *on* $\mathbb{R}$.

Proof. (1) $\Rightarrow$ (2). This follows immediately from Lemma 2.2.

(2) $\Rightarrow$ (1). Suppose $B \neq C$. Relabeling if necessary we may assume that there is a vector $\mathbf{x} = (x_0, ..., x_n)$ such that

$$\mathbf{x} \in B \backslash C.$$

To complete the proof of this implication, we need only show that $B_{\mathbf{t}} \neq C_{\mathbf{t}}$ for some nonzero $\mathbf{t} \in \mathbb{R}^n$. Since C is compact and convex and $\mathbf{x} \notin C$ there is a hyperplane that strictly separates C from $\mathbf{x}$ [Val, Theorem 2.10, p. 25]. Thus, there is a vector $(t_0, ..., t_n) \in \mathbb{R}^{n+1}$ and a real number β such that for any $\mathbf{y} = (y_0, ..., y_n) \in C$ we have

$$t_0 x_0 + \cdots + t_n x_n < \beta < t_0 y_0 + \cdots + t_n y_n.$$

Hence

$$(x_0, t_1 x_1 + \cdots + t_n x_n) \neq (y_0, t_1 y_1 + \cdots + t_n y_n).$$

In other words, with $\mathbf{t} = (t_1, ..., t_n)$ we have $\pi_{\mathbf{t}}(\mathbf{x}) \neq \pi_{\mathbf{t}}(\mathbf{y})$ for every $\mathbf{y} \in C$. Hence $\pi_{\mathbf{t}}(\mathbf{x}) \in B_{\mathbf{t}} \backslash C_{\mathbf{t}}$ and $B_{\mathbf{t}} \neq C_{\mathbf{t}}$.

(1) $\Rightarrow$ (3). Fixing s and $\mathbf{t}$, there is precisely one value of α such that $P(\mathbf{t}, s, \alpha)$ is a hyperplane of support for B and $B \subset P^{\uparrow}(\mathbf{t}, s, \alpha)$. Since $B = C$, α has the same properties with respect to C and so

$$P(\mathbf{t}, s, \alpha) \cap B = P(\mathbf{t}, s, \alpha) \cap C.$$

Hence, by part (3) of Theorem 2.3 we have

$$\Psi_M([p^-_{\mathbf{t},s}, p^+_{\mathbf{t},s}]) = P(\mathbf{t}, s, \alpha) \cap B = P(\mathbf{t}, s, \alpha) \cap C = \Psi_N([q^-_{\mathbf{t},s}, q^+_{\mathbf{t},s}])$$

and therefore we get that $\tau_M(p^{\pm}_{\mathbf{t},s}) = \tau_n(q^{\pm}_{\mathbf{t},s})$.

(3) ⇒ (5). Suppose (3) holds. In this case, (5) holds when f is the characteristic function of the interval $(-\infty, s)$ or $(-\infty, s]$, for any $s \in \mathbb{R}$. Since these intervals generate the Borel structure of $\mathbb{R}$, and τ_M and τ_N are normal and linear, (5) holds when f is the characteristic function of any Borel set. Since any bounded Borel function can be uniformly approximated by linear combinations of such characteristic functions, (5) holds in general.

(4) ⇒ (5). If (4) holds, then (5) holds if f is any polynomial. Since such polynomials are weakly dense in the von Neumann algebras generated by $b_{\mathbf{t}}$ and $c_{\mathbf{t}}$ and the traces are normal, we get that (5) holds for every bounded Borel function.

(5) ⇒ (3), (4). These implications are trivial.

(5) ⇒ (2). First take f to be the characteristic function of the interval $(-\infty, s)$ or $(-\infty, s]$ so that $f(b_{\mathbf{t}}) = p^{\pm}_{\mathbf{t},s}$. Applying (5) we get $\tau_M(p^{\pm}_{\mathbf{t},s}) = \tau_N(q^{\pm}_{\mathbf{t},s})$. Next, define the function g on $\mathbb{R}$ by the formula $g(t) = tf(t)$ so that $g(b_{\mathbf{t}}) = b_{\mathbf{t}} p^{\pm}_{\mathbf{t},s}$ and argue as above to conclude that $\tau_M(b_{\mathbf{t}} p^{\pm}_{\mathbf{t},s}) = \tau_N(c_{\mathbf{t}} q^{\pm}_{\mathbf{t},s})$. Thus we have

$$(\tau_M(p^{\pm}_{\mathbf{t},s}), \tau_M(b_{\mathbf{t}} p^{\pm}_{\mathbf{t},s})) = (\tau_N(q^{\pm}_{\mathbf{t},s}), \tau_N(c_{\mathbf{t}} q^{\pm}_{\mathbf{t},s}))$$

for every nonzero $\mathbf{t}$ and all real s. Since these are precisely the extreme points on the lower boundaries of $B_{\mathbf{t}}$ and $C_{\mathbf{t}}$ respectively by part (1) of Theorem 1.5, the lower boundaries coincide. Applying part (3) of Theorem 1.5, we see that the upper boundaries also coincide and therefore $B_{\mathbf{t}} = C_{\mathbf{t}}$. ∎

The following lemma, which is well known folklore, will be employed in the sequel.

Lemma 3.3. *The tracial representations of M and N are equivalent if and only if*

$$\tau_M(\phi(b_1, ..., b_n)) = \tau_N(\phi(c_1, ..., c_n))$$

for every monomial ϕ in n (commuting or noncommuting) variables.

Proof. If ϕ is a monomial, let us write $\phi(\mathbf{b})$ and $\phi(\mathbf{c})$ for $\phi(b_1, ..., b_n)$ and $\phi(c_1, ..., n_n)$ to simplify the notation. First, suppose that $\tau_M(\phi(\mathbf{b})) = \tau_N(\phi(\mathbf{c}))$ for every monomial ϕ and write

$$u\pi_M(\phi(\mathbf{b}))\, \xi_M = \pi_N(\phi(\mathbf{c}))\, \xi_N.$$

Next, extend this definition by linearity to the case where ϕ is a polynomial. If ϕ_1 and ϕ_2 are two such polynomials, then we have

$$\begin{aligned}
&(\pi_M(b_i)\, \pi_M(\phi_1(\mathbf{b}))\, \xi_M, \pi_M(\phi_2(\mathbf{b}))\, \xi_M) \\
&\quad = \tau_M(\phi_2(\mathbf{b})^*\, b_i \phi_1(\mathbf{b})) \\
&\quad = \tau_N(\phi_2(\mathbf{c})^*\, c_i \phi_1(\mathbf{c})) \\
&\quad = (\pi_N(c_i)\, \pi_N(\phi_1(\mathbf{c}))\, \xi_N, \pi_N(\phi_2(\mathbf{c}))\, \xi_N) \\
&\quad = (\pi_N(c_i)\, u\pi_M(\phi_1(\mathbf{b}))\, \xi_M, u\pi_M(\phi_2(\mathbf{b}))\, \xi_M).
\end{aligned}$$

Since such polynomials are dense in H_M and H_N, u extends to a unitary transformation from H_M to H_N with the desired properties.

Conversely, if the tracial representations of M and N are equivalent, then we have

$$\begin{aligned}
\tau_N(\phi(\mathbf{c})) &= (\pi_N(\phi(\mathbf{c}))\, \xi_N, \xi_N) \\
&= (u^*\pi_M(\phi(\mathbf{b}))\, u\xi_N, \xi_N) \\
&= (\pi_M(\phi(\mathbf{b}))\, \xi_M, \xi_M) \\
&= \tau_M(\phi(\mathbf{b})). \quad \blacksquare
\end{aligned}$$

Let us begin by considering the abelian case, which is relatively easy.

THEOREM 3.4. *If M and N are abelian and the spectral scales of $b_1, ..., b_n$ and $c_1, ..., c_n$ relative to τ_M and τ_N are equal, then the tracial representations of M and N are equivalent.*

Proof. By Lemma 3.3, it is enough to show that if $\phi_{k_1, ..., k_n}(x_1, ..., x_n) = x_1^{k_1} \cdots x_n^{k_n}$ denotes a monomial in the commuting variables $x_1, ..., x_n$, then

$$\tau_M(\phi_{k_1, ..., k_n}(b_1, ..., b_n)) = \tau_N(\phi_{k_1, ..., k_n}(c_1, ..., c_n)).$$

To show this, note that we have $\tau_M(b_\mathbf{t}^k) = \tau_N(c_\mathbf{t}^k)$ for each $k \in \mathbb{N}$ and each $\mathbf{t} \in \mathbb{R}^n$ by part (4) of Theorem 3.2. Fixing k, we have

$$b_\mathbf{t}^k = (t_1 b_1 + \cdots + t_n b_n)^k = \sum t_1^{k_1} \cdots t_n^{k_n} \phi_{k_1, ..., k_n}(b_1, ..., b_n),$$

where the sum is taken over all monomials $\phi_{k_1, ..., k_n}$ with $k_1 + \cdots + k_n = k$. Similarly we have $c_\mathbf{t}^k = \sum t_1^{k_1} \cdots t_n^{k_n} \phi_{k_1, ..., k_n}(c_1, ..., c_n)$. Applying the trace yields

$$\sum t_1^{k_1} \cdots t_n^{k_n} \tau_M(\phi_{k_1, \ldots, k_n}(b_1, \ldots, b_n))$$
$$= \sum t_1^{k_1} \cdots t_n^{k_n} \tau_N(\phi_{k_1, \ldots, k_n}(c_1, \ldots, c_n)).$$

Since these polynomials in $t_1, \ldots, t_n$ are equal, we may equate coefficients. Thus $\tau_M(\phi_{k_1, \ldots, k_n}(b_1, \ldots, b_n)) = \tau_N(\phi_{k_1, \ldots, k_n}(c_1, \ldots, c_n))$ for every monomial of total degree k, as desired. ∎

As noted above, it turns out that equality of the spectral scales is *not* sufficient to guarantee that the n-tuples are unitarily equivalent in their tracial representations in the general noncommutative case. The following simple 3×3 matrix example shows what can go wrong.

EXAMPLE 3.5. Write

$$b_1 = c_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}, \quad \text{and} \quad c_2 = \begin{bmatrix} 0 & x' & y' \\ x' & 0 & z' \\ y' & z' & 0 \end{bmatrix},$$

where x, y, z, x', y', and z' are positive real numbers. It is clear that

$$\{b_1, b_2\}'' = M = \mathrm{Mat}_3(\mathbb{C}) \qquad \text{and} \qquad \{c_1, c_2\}'' = N = \mathrm{Mat}_3(\mathbb{C}).$$

We have $B = C$ if and only if $B_{\mathbf{t}} = C_{\mathbf{t}}$ for every vector $\mathbf{t}$ in $\mathbb{R}^n$ by part (2) of Theorem 3.2. Moreover, $B_{\mathbf{t}} = C_{\mathbf{t}}$ if and only if the spectra of $b_{\mathbf{t}}$ and $c_{\mathbf{t}}$ are the same and the traces of corresponding spectral projections are equal by part (3) of Theorem 3.2. This amounts to requiring that $b_{\mathbf{t}}$ and $c_{\mathbf{t}}$ have the same eigenvalues with the same multiplicities and this occurs if and only if the characteristic polynomials of $b_{\mathbf{t}}$ and $c_{\mathbf{t}}$ are equal.

It is easy to check that we get this equality if and only if x, y, z, x', y', and z' satisfy conditions

$$\begin{gathered} x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \\ xyz = x'y'z' \\ 3x^2 + 2y^2 + z^2 = 3x'^2 + 2y'^2 + z'^2. \end{gathered} \tag{$*$}$$

Next, it is straightforward to calculate that

$$\tau_M(b_1 b_2 b_1 b_2) = 4x^2 + 6y^2 + 12z^2.$$

Hence, we have $\tau_M(b_1 b_2 b_1 b_2) = \tau_N(c_1 c_2 c_1 c_2)$ if and only if

$$4x^2 + 6y^2 + 12z^2 = 4x'^2 + 6y'^2 + 12z'^2. \tag{$**$}$$

Further, since the determinant of

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 6 & 12 \end{bmatrix}$$

is -4, if both $(*)$ and $(**)$ hold, then we get $\tau_M(b_1 b_2 b_1 b_2) = \tau_N(c_1 c_2 c_1 c_2)$ if and only if

$$x^2 = x'^2, \qquad y^2 = y'^2 \qquad \text{and} \qquad z^2 = z'^2.$$

Since all entries are positive, it follows from these conditions that $x = x'$, $y = y'$, and $z = z'$.

Thus if we can find x, y, z, x', y' and z' such that $(\,*\,)$ holds, but $(x, y, z) \neq (x', y', z')$, then we will have $B_{\mathbf{t}} = C_{\mathbf{t}}$ for every nonzero vector $\mathbf{t}$ (so that $B = C$ by Theorem 3.2), but that $\tau_M(b_1 b_2 b_1 b_2) \neq \tau_N(c_1 c_2 c_1 c_2)$ (so that the n-tuples are not unitarily equivalent in their tracial representations, by Lemma 3.3). It is easy to check that these conditions are met if we take

$$x = 3, \qquad y = 2, \qquad z = 1$$

and

$$x' = \sqrt{5 + 3\sqrt{3}}, \qquad y' = \sqrt{12 - 6\sqrt{3}}, \qquad z' = \sqrt{3\sqrt{3} - 3}.$$

Thus, the nonabelian analogue of Theorem 3.4 is false. If the spectral scales of $b_1, ..., b_n$ and $c_1, ..., c_n$ are equal, the tracial representations of M and N need not be equivalent.

Our remaining goal in this section is to show that such an equivalence does hold if the spectral scales are "completely" equal. That is, they are equal when we pass to the matrix analogue of this notion.

We shall find it convenient to view $\Psi_M(a)$ as a diagonal matrix in $\mathrm{Mat}_{n+1}(\mathbb{C})$. Thus we have

$$\Psi_M(a) = \begin{bmatrix} \tau_M(a) & 0 & \cdots & 0 \\ 0 & \tau_M(ab_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau_M(ab_n) \end{bmatrix}.$$

DEFINITION 3.6. For each positive integer m define

$$\Psi_M^{(m)}\colon \mathrm{Mat}_m(M) \to \mathrm{Mat}_{m(n+1)}(\mathbb{C})$$

by the formula

$$\Psi_M^{(m)}([a_{ij}]) = [\Psi_M(a_{ij})]$$

and write

$$B^{(m)} = \Psi^{(m)}(\mathrm{Mat}_m(M)_1^+).$$

Observe that elements in $B^{(m)}$ are self-adjoint matrices. Let $\{C^{(m)}\}$ denote the corresponding sequence of sets determined by N and Ψ_N.

We call the sequence $\{B^{(m)}\}$ the *complete spectral scale* of $b_1, ..., b_n$ *relative to* τ_M.

Note that $B^{(1)} = B$ is the usual spectral scale. The use of matrix levels in the noncommutative setting is increasingly being recognized as appropriate and useful; see [E], for example. In fact, the complete spectral scale is an example of a matrix convex set as defined in [W, EW].

It will be shown in the final theorem of this section that the tracial representations of M and N are equivalent if and only if the corresponding complete spectral scales are equal. It is convenient to present the proof in a series of Lemmas.

Given noncommuting variables $x_1, ..., x_n$ and nonnegative integers $k_1, ..., k_n$, we now write $\phi = \phi_{k_1, ..., k_n}(x_1, ..., x_n)$ for the sum of all monomials in which x_i appears k_i times. We call ϕ a *cycle polynomial.*

Throughout the remainder of this section, we use the notation developed above. Thus, B and C are the spectral scales of the n-tuples $b_1, ..., b_n$ and $c_1, ..., c_n$ and $\{B^{(m)}\}$ and $\{C^{(m)}\}$ denote the respective complete spectral scales relative to τ_M and τ_N.

LEMMA 3.7. *If* $B = B^{(1)} = C^{(1)} = C$, *then*

$$\tau_M(\phi(b_1, ..., b_n)) = \tau_N(\phi(c_1, ..., c_n))$$

for every circle polynomial ϕ *in n variables.*

Proof. This is established by repeating virtually verbatim the second paragraph in the proof of Theorem 3.4. The only difference is that $\phi_{k_1, ..., k_n}$ now stands for a cycle polynomial. ∎

Recall that if $\mathbf{t} = (t_1, ..., t_n)$ is a vector in $\mathbb{R}^n$, then $b_{\mathbf{t}} = t_1 b_1 + \cdots + t_n b_n$. We now extend this notation to include the case where the coefficients are *complex*. Thus we write $b_{\mathbf{t}} = t_1 b_1 + \cdots + t_n b_n$ for $\mathbf{t} \in \mathbb{C}^n$.

LEMMA 3.8. *If* $B = C$ *then* $\tau_M(b_{\mathbf{t}}^m) = \tau_N(c_{\mathbf{t}}^m)$ *for every* $m \in \mathbb{N}$ *and* $\mathbf{t} \in \mathbb{C}^n$.

Proof. Observe that

$$\mathbf{b}_{\mathbf{t}}^m = \sum_{m_1 + \cdots + m_n = m} t_1^{m_1} \cdots t_n^{m_n} \phi_{m_1, ..., m_n}(b_1, ..., b_n),$$

where $\phi_{m_1, ..., m_n}$ is a cycle polynomial. Moreover, we have

$$\tau_M(\phi_{m_1, ..., m_n}(b_1, ..., b_n)) = \tau_N(\phi_{m_1, ..., m_n}(c_1, ..., c_n))$$

for every cycle polynomial by Lemma 3.7. Hence

$$\begin{aligned}\tau_M(b_{\mathbf{t}}^m) &= \sum_{m_1 + \cdots + m_n = m} t_1^{m_1} \cdots t_n^{m_n} \tau_M(\phi_{m_1, ..., m_n}(b_1, ..., b_n)) \\ &= \sum_{m_1 + \cdots + m_n = m} t_1^{m_1} \cdots t_n^{m_n} \tau_N(\phi_{m_1, ..., m_n}(c_1, ..., c_n)) \\ &= \tau_N(c_{\mathbf{t}}^m),\end{aligned}$$

as desired. ∎

Let $\tau_M^{(m)}$ stand for the normalized trace on $\mathrm{Mat}_m(M)$. Thus, if $\mathbf{a} = [a_{ij}]$ is in $\mathrm{Mat}_m(M)$, then

$$\tau_M^{(m)}(\mathbf{a}) = \frac{1}{m}(\tau_M(a_{11}) + \cdots + \tau_M(a_{mm})).$$

Write $\tau_N^{(m)}$ for the corresponding trace on $\mathrm{Mat}_m(N)$.

We now wish to define certain self-adjoint elements of $\mathrm{Mat}_m(M)$. Let $\mathbf{v}_{i, j, k}$ denote the $m \times m$ matrix whose (i, j)-entry is b_k and whose other entries are all zero. Now write (where $\imath$ denotes $\sqrt{-1}$)

$$\mathbf{b}_{i, j, k} = \begin{cases} \mathbf{v}_{i, i, k} & \text{if } \; i = j, \\ \mathbf{v}_{i, j, k} + \mathbf{v}_{j, i, k} & \text{if } \; i < j, \quad 1 \leqslant i, \quad j \leqslant m, \quad 1 \leqslant k \leqslant n. \\ \imath \mathbf{v}_{i, j, k} - \imath \mathbf{v}_{j, i, k} & \text{if } \; i > j, \end{cases}$$

Observe that each $\mathbf{b}_{i, j, k}$ is self-adjoint and has at most two nonzero entries. Also, there are $m^2 n$ such matrices. Write $\mathbf{c}_{i, j, k}$ for the corresponding elements of $\mathrm{Mat}_m(N)$.

We now use the $\mathbf{b}_{i, j, k}$'s and $\mathbf{c}_{i, j, k}$'s to define two auxiliary spectral scales that will be useful. For $a \in \mathrm{Mat}_m(M)$, write

$$\Phi_M^{(m)}(\mathbf{a}) = (\tau_M^{(m)}(\mathbf{a}), \tau_M^{(m)}(\mathbf{b}_{1, 1, 1}\mathbf{a}), ..., \tau_M^{(m)}(\mathbf{b}_{i, j, k}\mathbf{a}), ..., \tau_M^{(m)}(\mathbf{b}_{m, m, n}\mathbf{a})),$$

where the indices $\{i, j, k\}$ are ordered lexicographically and let $B_\Phi^{(m)}$ denote the spectral scale of $\{\mathbf{b}_{i, j, k} : 1 \leqslant i, j \leqslant m, 1 \leqslant k \leqslant n\}$ relative to $\tau_M^{(m)}$. Finally, define $\Phi_N^{(m)}$ and $C_\Phi^{(m)}$ analogously.

Lemma 3.9. *The complete spectral scales $\{B^{(m)}\}$ and $\{C^{(m)}\}$ are equal if and only if $B_\Phi^{(m)} = C_\Phi^{(m)}$ for every positive integer m.*

Proof. Fix $\mathbf{a} = [a_{ij}]$ in $\mathrm{Mat}_m(M)_1^+$. The (i, j)-entry of $\Psi_M^{(m)}(\mathbf{a})$ has the form

$$\begin{bmatrix} \tau_M(a_{ij}) & 0 & \cdots & 0 \\ 0 & \tau_M(b_1 a_{ij}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau_M(b_n a_{ij}) \end{bmatrix}.$$

On the other hand we have

$$\tau_M^{(m)}(\mathbf{b}_{i,\,j,\,k}\mathbf{a}) = \begin{cases} \dfrac{1}{m}\,\tau_M(b_k a_{ii}) & \text{if} \quad i = j \\ \dfrac{1}{m}\,\tau_M(b_k(a_{ij} + a_{ji})) & \text{if} \quad i < j \\ \dfrac{1}{m}\,\tau_M(b_k(\imath a_{ij} - \imath a_{ji})) & \text{if} \quad i > j. \end{cases}$$

Thus, each $\tau_M^{(m)}(b_{i,\,j,\,k}\mathbf{a})$ is completely determined by the values $\tau_M(b_k a_{ij})$ for $1 \leqslant i, j \leqslant m$ and conversely, each $\tau_M(b_k a_{ij})$ is determined by the values of $\tau_M^{(m)}(\mathbf{b}_{i,\,j,\,k}\mathbf{a})$.

If $B^{(m)} = C^{(m)}$, then there is $\mathbf{a}' = [a'_{ij}]$ in $\mathrm{Mat}_m(N)_1^+$ such that $\Psi_M^{(m)}(\mathbf{a}) = \Psi_N^{(m)}(\mathbf{a}')$. With this, we get that $\tau_M(a_{ij}) = \tau_N(a'_{ij})$ for $1 \leqslant i, j \leqslant m$ and $\tau_M(b_k a_{ij}) = \tau_N(c_k a'_{ij})$ for $1 \leqslant i, j \leqslant m$, $1 \leqslant k \leqslant n$. Hence $\tau_M(\mathbf{b}_{i,\,j,\,k} a_{ij}) = \tau_N(\mathbf{c}_{i,\,j,\,k} a'_{ij})$ for each index triple i, j, k and therefore $B_\Phi^{(m)} = C_\Phi^{(m)}$. The proof of the converse is similar. ∎

Next we want to consider complex linear combinations of the $\mathbf{b}_{i,\,j,\,k}$'s and the $\mathbf{c}_{i,\,j,\,k}$'s. If $\mathbf{t} = (t_{i,\,j,\,k}) \in \mathbb{C}^{m^2 n}$, let us write

$$\mathbf{b}_\mathbf{t} = \sum t_{i,\,j,\,k}\mathbf{b}_{i,\,j,\,k} \qquad \text{and} \qquad \mathbf{c}_\mathbf{t} = \sum t_{i,\,j,\,k}\mathbf{c}_{i,\,j,\,k}.$$

LEMMA 3.10. *If the complete spectral scales $\{B^{(m)}\}$ and $\{C^{(m)}\}$ are equal, then*

$$\tau_M(\phi(b_1, ..., b_n)) = \tau_N(\phi(c_1, ..., c_n))$$

for every monomial ϕ in n noncommuting variables.

Proof. Write $\phi(b_1, ..., b_n) = b_{l_1} \cdots b_{l_m}$ and let $\mathbf{b}_{l_1, ..., l_m}$ denote the matrix whose first upper diagonal consists of $b_{l_1}, b_{l_2}, ..., b_{l_{m-1}}$, the $(m, 1)$-entry is b_{l_m} and all other entries are zero. Thus, we have

$$\mathbf{b}_{l_1, \dots, l_m} = \begin{bmatrix} 0 & b_{l_1} & & \\ & \ddots & \ddots & \\ & & \ddots & b_{l_{m-1}} \\ b_{l_m} & & & 0 \end{bmatrix}.$$

Similarly, we have $\phi(c_1, \dots, c_n) = c_{l_1} \cdots c_{l_m}$ and we may define $\mathbf{c}_{l_1, \dots, l_m}$ analogously.

Next define $\mathbf{t} = (t_{i,j,k})$ by the formulas

$$t_{i,i+1,l_i} = \frac{1}{2}, \qquad 1 \leqslant i < m,$$

$$t_{i+1,i,l_i} = -\frac{\imath}{2}, \qquad 1 \leqslant i < m,$$

$$t_{1,m,l_m} = \frac{1}{2},$$

$$t_{m,1,l_m} = -\frac{\imath}{2},$$

$$t_{i,j,k} = 0, \qquad \text{otherwise},$$

and observe that $\mathbf{t} \in \mathbb{C}^{m^2 n}$ because $1 \leqslant l_j \leqslant n$ for each j. With these choices we get $\mathbf{b}_\mathbf{t} = \mathbf{b}_{l_1, \dots, l_m}$ and $\mathbf{c}_\mathbf{t} = \mathbf{c}_{l_1, \dots, l_m}$.

Next, note that since $\{B^{(m)}\} = \{C^{(m)}\}$, we get that the spectral scales $B^{(m)}_\Phi$ and $C^{(m)}_\Phi$ are equal by Lemma 3.9 and therefore

$$\tau^{(m)}_M(\mathbf{b}^m_\mathbf{t}) = \tau^{(m)}_N(\mathbf{c}^m_\mathbf{t})$$

by Lemma 3.8. Hence, we have

$$\tau^{(m)}_M((\mathbf{b}_{l_1, \dots, l_m})^m) = \tau^{(m)}_M(\mathbf{b}^m_\mathbf{t}) = \tau^{(m)}_N(\mathbf{c}^m_\mathbf{t}) = \tau^{(m)}_N((\mathbf{c}_{l_1, \dots, l_m})^m).$$

Next, we have

$$(\mathbf{b}_{l_1, \dots, l_m})^m = \begin{bmatrix} 0 & b_{l_1} & & \\ & \ddots & \ddots & \\ & & \ddots & b_{l_{m-1}} \\ b_{l_m} & & & 0 \end{bmatrix}^m$$

$$= \begin{bmatrix} b_{l_1} b_{l_2} \cdots b_{l_m} & 0 & \cdots & 0 \\ 0 & b_{l_2} b_{l_3} \cdots b_{l_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{l_m} b_{l_1} \cdots b_{l_{m-1}} \end{bmatrix}$$

and so

$$\tau_M^{(m)}((\mathbf{b}_{l_1, ..., l_m})^m) = \frac{1}{m}(\tau_M(b_{l_1} \cdots b_{l_m}) + \cdots + \tau_M(b_{l_m} b_{l_1} \cdots b_{l_{m-1}}))$$
$$= \tau_M(b_{l_1} b_{l_2} \cdots b_{l_m}) = \tau_M(\phi(b_1, ..., b_n)).$$

Similarly, we have

$$\tau_M^{(m)}((\mathbf{c}_{l_1, ..., l_m})^m) = \tau_M(c_{l_1} \cdots c_{l_m}) = \tau_N(\phi(c_1, ..., c_n))$$

and therefore

$$\tau_M(\phi(b_1, ..., b_n)) = \tau_M^{(m)}((\mathbf{b}_{l_1, ..., l_m})^m) = \tau_N^{(m)}((\mathbf{c}_{l_1, ..., l_m})^m)$$
$$= \tau_N(\phi(c_1, ..., c_n)). \quad \blacksquare$$

THEOREM 3.11. *The tracial representations of M and N are equivalent if and only if their complete spectral scales $\{B^{(m)}\}$ and $\{C^{(m)}\}$ relative to τ_M and τ_N are equal.*

Proof. First suppose $\{B^{(m)}\} = \{C^{(m)}\}$. In this case we have $\tau_M(\phi(b_1, ..., b_n)) = \tau_N(\phi(c_1, ..., c_n))$ for every monomial ϕ by Lemma 3.10. Hence, the tracial representations of M and N are equivalent by Lemma 3.3.

Conversely, if the tracial representations are equivalent, then the implementing unitary u induces corresponding equivalences at all matrix levels. Hence, the complete spectral scales agree. $\blacksquare$

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