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An Improvement in The Superposition Theorem of Kolmogorov

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Designate by E^n the n-fold cartesian product of closed unit intervals $E = [0, 1]$, and let $C(E^n)$ stand for the space of real-valued continuous functions defined on E^n.

The following theorem is due to Kolmogorov [1]:

THEOREM 1. *For each integer $n \geqslant 2$ there exist monotonic increasing functions $\psi^{pq} \in C(E)$ with the property that each function $f \in C(E^n)$ can be represented in the form*

$$f(x_1, \ldots, x_n) = \sum_{q=0}^{2n} g_q \left[\sum_{p=1}^{n} \psi^{pq}(x_p) \right]$$

with continuous functions g_q.

Fridman [2] proved that this theorem is possible with monotonic functions ψ^{pq} belonging to the Lipschitz class Lip (1)[1]. In this note we prove that Kolmogorov's theorem is possible when the functions ψ^{pq} are replaced by translations of a single function $\psi \in \mathrm{Lip}\,(1)$. Specifically, the following stronger version is proved:

THEOREM 2. *For each integer $n \geqslant 2$ there is a constant $\lambda \neq 0$ and a single monotonic function $\psi \in \mathrm{Lip}\,(1)$ having the property that each function $f \in C(E^n)$ can be represented in the form*

$$f(x_1, \ldots, x_n) = \sum_{q=0}^{2n} g_q \left[\sum_{p=1}^{n} \lambda^{p-1} \psi(x_p + q\epsilon) \right],$$

where the functions g_q are continuous and $\epsilon \neq 0$ is any constant.

This theorem improves the author's earlier result [3] which established Theorem 2 for a function $\psi \in \mathrm{Lip}[\ln 2/\ln(2n + 2)]$.

[1] $\varphi \in \mathrm{Lip}(\alpha)$ if there are constants $0 < \alpha \leqslant 1$ and A such that $| \varphi(t_1) - \varphi(t_2)| \leqslant A \,| t_1 - t_2 |^\alpha$ for all points t_1 and t_2 in the domain of φ.

In what follows, $S_r^q(i_{1r}, ..., i_{nr})$ will designate for each positive integer r the cartesian product of closed intervals $E_r^q(i_{pr})$, laid on the p-th coordinate axis in Euclidean space R^n, the indices i_{pr} having certain domains $1 \leqslant i_{pr} \leqslant m_r$, where $m_r \to \infty$ as $r \to \infty$. The following lemma was proved by Kolmogorov in [1]:

LEMMA 1. *Let $S_r^q(i_{1r}, ..., i_{nr})$ be cubes in R^n with the following properties:*

(1) $S_r^q(i_{1r}, ..., i_{nr}) \cap S_r^q(i'_{1r}, ..., i'_{nr}) = \varnothing$ *for all values of q and r when*

$$(i_{1r}, ..., i_{nr}) \neq (i'_{1r}, ..., i'_{nr});$$

(2) *For each value of r, each point of E^n is contained in at least $n + 1$ cubes $S_r^q(i_{1r}, ..., i_{nr})$;*

(3) $\mathrm{Diam}[S_r^q(i_{1r}, ..., i_{nr})] \to 0$ *as $r \to \infty$ uniformly in $(i_{1r}, ..., i_{nr})$*, diam (S) *standing for the diameter of S.*

Let these be continuous functions $\psi_q \in C(E^n)$ such that

(4) $\psi_q[S_r^q(i_{1r}, ..., i_{nr})] \cap \psi_q[S_r^q(i'_{1r}, ..., i'_{nr})] = \varnothing$ *whenever*

$$(i_{1r}, ..., i_{nr}) \neq (i'_{1r}, ..., i'_{nr}),$$

$\psi_q(S)$ being the image of the set S under ψ_q.

Then each function $f \in C(E^n)$ has a representation of the form

$$f(x_1, ..., x_n) = \sum_{q=0}^{2n} g_q[\psi_q(x_1, ..., x_n)]$$

with continuous functions q_q.

In additions, Kolmogorov showed how to construct appropriate cubes and functions $\psi_q(x_1, ..., x_n)$ (of the type stated in Theorem 1) which meet the demands in Lemma 1. A detailed construction of this kind may be found in [3]. To prove Theorem 2 it thus suffices to show that there are functions

$$\psi_q(x_1, ..., x_n) = \sum_{p=1}^{n} \lambda^{p-1}\psi(x_p + q\epsilon), \qquad \psi \in \mathrm{Lip}(1)$$

which satisfy Lemma 1 for suitable cubes $S_r^q(i_{1r}, ..., i_{nr})$. This is the purpose of the following two lemmas.

LEMMA 2. *Let $\epsilon \neq 0$ be a given number. For each natural number r there is a system of intervals $E_r(i_r)$, $1 \leqslant i_r \leqslant m_r$, $m_r \to \infty$ as $r \to \infty$, having the following properties: Put*

$$E_r^q(i_r) = \begin{cases} E_r(i_r) & \text{when} \quad q = 0, \\ \{x \mid x + q\epsilon \in E_r(i_r)\} & \text{when} \quad 1 \leqslant q \leqslant 2n, \end{cases}$$

and let $\{\delta_r\}$ *be a preassigned null-sequence of positive numbers. Then,*

(5) $E_r^q(i_r) \cap E_r^q(i_r') = \varnothing$ *for all values of* q *and* r *when* $i_r \neq i_r'$.

(6) *For each value of* r, *each point of* E *is contained in at least* $2n$ *intervals* $E_r^q(i_r)$.

(7) *Let* q' *be given,* $0 \leqslant q' \leqslant 2n$. *If* $s > r$ *and* $E_s^{q'}(i_s) \not\subset E_r^q(i_r)$ *for* $1 \leqslant i_r \leqslant m_r$, *then* $\operatorname{diam}[E_s^{q'}(i_s)] \leqslant \delta_s$ *for all values of* q *for which this is true.*

(8) $\operatorname{Diam}[E_r^q(i_r)] \to 0$ *as* $r \to \infty$ *uniformly in* i_r.

LEMMA 3. *There is a monotonic function* $\psi \in \mathrm{Lip}(1)$ *and a constant* $\lambda \neq 0$ *with the property that if*

$$\psi(x_1, \ldots, x_n) = \sum_{p=1}^{n} \lambda^{p-1}\psi(x_p),$$

then

$$\psi[E_r(i_r)] \cap \psi[E_r(i_r')] = \varnothing \qquad \textit{whenever} \qquad i_r \neq i_r'.$$

Designating for each value of q and r the n-fold cartesian product of intervals $E_r^q(i_r)$ by $S_r^q(i_{1r}, \ldots, i_{nr})$, we find that

$$S_r^q(i_{1r}, \ldots, i_{nr}) = \{(x_1, \ldots, x_n) \mid (x_1 + q\epsilon, \ldots, x_n + q\epsilon) \in S_r^0(i_{1r}, \ldots, i_{nr})\}.$$

It is now easy to show that properties (1)–(3) of Lemma 1 are implied by properties (5), (6), and (8) of Lemma 2. Property (4) evidently follows from Lemma 3, and it thus remains to prove the last two lemmas. We shall outline the proofs here.

Proof of Lemma 2. This lemma is proved by induction on r; we prove it for a constant

$$0 < \epsilon \leqslant \frac{1}{2n}.$$

For $r = 1$ we let

$$E_1^q(i_1) = [-q\epsilon, 2 - q\epsilon], \qquad i_1 = 1, \qquad 0 \leqslant q \leqslant 2n.$$

Now suppose that the families of intervals $\{E_{r-1}^q(i_{r-1})\}$, $0 \leqslant q \leqslant 2n$, are already determined. We emphasize that these families are congruent to one another, i.e., they are translations of the family $\{E_{r-1}^0(i_{r-1})\}$. In the family $\{E_{r-1}^0(i_{r-1})\}$ we fix an interval $E_{s-1}^0(i_{s-1})$ such that

$$\operatorname{diam}[E_{s-1}^0(i_{s-1})] \geqslant \operatorname{diam}[E_{r-1}^0(i_{r-1})]$$

for all admitted values of i_{r-1}. Let the midpoint of this interval be ξ_{s-1}^0; then $\xi_{s-1}^0 - q\epsilon \in E_{s-1}^0(i_{s-1})$ for $0 \leqslant q \leqslant 2n$. We delete the open interval

$$G_{s-1}^q(i_{s-1}) = (\xi_{s-1}^0 - q\epsilon - \delta_s, \xi_{s-1}^0 - q\epsilon + \delta_s)$$

from $E_{s-1}^q(i_{s-1})$ for $0 \leqslant q \leqslant 2n$, δ_s being a positive number of length not exceeding $\frac{1}{3}$ the width of the narrowest gap between successive intervals $E_{r-1}^0(i_{r-1})$ but one whose exact value is to be determined when we prove Lemma 3. From the family $\{E_{r-1}^0(i_{r-1})\}$ and the intervals

$$H_{s-1}^{q+t}(i_{x-1}) = [\xi_{s-1}^0 - (q+t)\epsilon - \delta_s, \xi_{s-1}^0 - (q+t)\epsilon + \delta_s]$$
$$-2n \leqslant t \leqslant 2n, \qquad t \neq 0$$

we construct for each q a new family $\{E_r{}^q(i_r)\}$ in the following way:

(i) If

$$H_{s-1}^{q+t}(i_{s-1}) \cap E_{r-1}^q(i_{r-1}) \neq \varnothing$$

for some t and i_{r-1}, we designate the union of these sets by $E_r{}^q(i_r)$ for a suitable value of i_r. Because of the choice of δ_s, there is at the most one value of i_{r-1} for each q for which this intersection is nonempty. It may happen, of course, that

$$E_{r-1}(i_{r-1}) \supset E_{s-1}^{q+t}(i_{s-1}),$$

in which case

$$E_{r-1}^q(i_{r-1}) = E_r{}^q(i_r).$$

(ii) If

$$H_{s-1}^{q+t}(i_{s-1}) \cap E_{r-1}^q(i_{r-1}) = \varnothing$$

for all t and i_{r-1}, then each set $H_{s-1}^{q+t}(i_{s-1})$ and $E_{r-1}^q(i_{r-1})$ gets redesignated as $E_r{}^q(i_r)$ for suitable values of i_r.

The families $\{E_r{}^q(i_r)\}$ so constructed can now be shown to satisfy Lemma 2.

Proof of Lemma 3. Pick a number $\lambda \neq 0$ which satisfies no equation

$$\sum_{p=1}^{n} a_p \lambda^{p-1} = 0$$

with rational coefficient a_p not all zero. The function ψ is obtained as a limit, $\psi = \lim_{r\to\infty} \psi_r$, where the functions ψ_r are constructed inductively to satisfy the following conditions:

(iii) For each r, ψ_r is continuous and nondecreasing, equals a rational constant on each interval $E_r(i_r)$, and is linear on the complements of these intervals.

(iv) For each r,

$$|\psi_r(x) - \psi_r(x')| < |x - x'|.$$

(v) $\psi_r[E_s(i_s)] \cap \psi_r[E_s(i_s')] = \varnothing$ when $i_s \neq i_s'$ and $s = 1, 2, \ldots, r$.

(vi) $\| \psi_r - \psi_{r-1} \| \leqslant 1/2^r$ for each r, the norm being the uniform norm.

Now, for $r = 1$ let $\psi_1 \equiv 0$. Suppose that ψ_{r-1} is already determined. Then ψ_r is constructed as follows: According to the construction of the intervals $E_r{}^0(i_r)$, there is one of the four possibilities for each of them:

(vii) $E_r{}^0(i_r) = E^0_{r-1}(i_{r-1})$ for some i_{r-1}.

(viii) $E_r{}^0(i_r) = E^0_{r-1}(i_{r-1}) \cup H^{0+t}_{s-1}(i_{s-1})$ for some i_{r-1}, t_1 and i_{s-1}, and

$$H^{0+t}_{s-1}(i_{s-1}) \not\subset E^0_{r-1}(i_{r-1}).$$

(ix) $E_r{}^0(i_r) = H^{0+t}_{s-1}(i_{s-1})$ for some t and i_{s-1}.

(x) $E_r{}^0(i_r)$ and $E_r{}^0(i_{r+1})$ intersect $E^0_{r-1}(i_{r-1})$ for some i_{r-1}.

(a) We let

$$\psi_r(x) = \psi_{r-1}(x) \qquad \text{for } x \in E_r{}^0(i_r)$$

when $E_r{}^0(i_r)$ is as in (vii) or (viii); in the latter case, we chose δ_s so small that (vi) will be satisfied. To construct ψ_r for an interval in (ix), let

$$E_r{}^0(i_r - 1) = [a_{i_r-1}, b_{i_r-1}], \qquad E_r{}^0(i_r) = [a_{i_r}, b_{i_r}],$$
$$E_r{}^0(i_r + 1) = [a_{i_r+1}, b_{i_r+1}].$$

(b) Select a rational number $\eta > 0$ such that

$$\psi_r(x) = \psi_{r-1}(x) + \eta < \psi_{r-1}(a_{i_r+1}) \qquad \text{for } x \in E_r{}^0(i_r).$$

Next, select δ_s so small that the line segments joining $\psi_{r-1}(b_{i_r-1})$ to $\psi_r(a_{i_r})$ and $\psi_r(b_{i_r})$ to $\psi_{r-1}(a_{i_r+1})$ still have slope less than 1. If necessary, we take for δ_s the smaller of the numbers fixed in (a) or (b).

(c) In the case (x) we let $\psi_r(x) = \psi_{r-1}(x)$ for $x \in E_r{}^0(i_r)$, and we again put $\psi_r(x) = \psi_{r-1}(x) + \rho$ for $x \in E_r{}^0(i_r + 1)$, where $\rho > 0$ is a rational number selected such that ψ_r is monotonic, and the line segment joining $\psi_r(b_{i_r})$ to $\psi_r(a_{i_r+1})$ has slope <1.

To complete the construction of ψ_r, we form a polygonal arc by joining with line segments the horizontal segments constructed above. This construction satisfies properties (iii)–(vi). Because ψ_r is rational on all intervals $E_s{}^0(i_s)$ for $1 \leqslant s \leqslant r$, it follows from the choice of λ and a simple argument that the lemma is satisfied.

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