REPRESENTATION OF CONTINUOUS FUNCTIONS OF THREE VARIABLES BY THE SUPERPOSITION OF CONTINUOUS FUNCTIONS OF TWO VARIABLES*

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Contents

Introduction

The present work is devoted to the proof of the following theorem, which was stated in an earlier note [1].

Theorem 1. *Every real continuous function* $f(x_1, x_2, x_3)$ *of three variables, defined on the unit cube* E^3, *can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3],$$

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Corrections in Mat. Sb. (n.S.) 56 (98):3 (1962), 392

where h_{ij} *and* φ_{ij} *are real continuous functions of two variables.*

For the proof of this theorem in note [1], use was made of two theorems whose complete proofs were not given in that paper. Here are these theorems.

Theorem 2. *Every continuous function* $f(x_1, x_2, x_3)$ *defined on* E^3 *can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1, x_2), x_3],$$

where h_i *and* φ_i *are continuous functions; the functions* h_i *are real and are defined on the product* $\Xi \times E^1$ *of the tree* (see [3], Chapter X) Ξ *by the interval* E^1, *while the functions* $\varphi_i(x_1, x_2)$ *are defined on a square and have for their values points of* Ξ. *Here* Ξ *is a tree, whose points have a branching index not greater than 3.*

Theorem 3. *Let* F *be any family of real, equi-continuous functions* $f(\xi)$ *defined on the tree* Ξ *all of whose points have a branching index* $\leqslant 3$. *Then one can realize the tree in the form of its homeomorphic image* X, *a subset of the three-dimensional unit cube* E^3, *in such a way that every function* f *of the family* F *can be represented in the form*

$$f(x) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1, x_2, x_3)$ *is the image in* X *of the element* $\xi \in \Xi$, $f(x) = f(\xi)$, *and the* $f_k(x_k)$ *are continuous real functions of one variable. Here* f_k *depends continuously on* f *in the sense of uniform convergence.*

Theorem 2 (with the exclusion of the last phrase) is contained in a work of A.N. Kolmogorov [2]. Its proof is also outlined there, but the proofs of the lemmas used there were not published. In Part I of the present work there are presented the proofs of these lemmas for the case when the branching index of the points of the obtained tree is not greater than 3. After that, the Theorem 2 given above is derived from these lemmas.

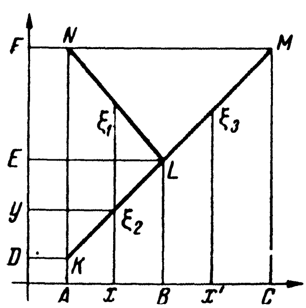


Figure 1. Representation in the form $\varphi(x) + \psi(y)$ of a function given on a Y-type tree.

For greater explicitness, let us consider the case $n = 2$ of the lemmas of the note [2]. The proofs (as well as the formulations) of these lemmas are somewhat different from those given by A.N. Kolmogorov. This is due to the introduction of the items 6) and 7) into the fundamental lemma, and to our desire to obtain

Theorem 2 in the formulation given above.

Theorem 3 is proved in the second part of this work. The ideas behind this theorem are quite simple.

Let a continuous function $f(\xi)$ $(\xi \in \Delta)$ be given on a Y-type tree (Figure 1). Then there exist continuous functions $f_1(x)$ and $f_2(y)$ such that $f_1(x) + f_2(y) = f(\xi)$ if x and y are the coordinates of the point $\xi \in \Delta$.

The proof can be accomplished, for example, as follows.

Suppose that the function $f_1(x)$ on AB is equal to $f(\xi_1)$ for a point $\xi_1 \in LN$ whose abscissa is x. In order that $f = f_1 + f_2$ on KL, one has to define $f_2(y)$ on DE as $f_2(y) = f(\xi_2) - f_1(x)$, where $\xi_2 \in KL$ is the point with coordinates x, y. Hereby, $f_2 = 0$ at the point E. Let $f_2(y) = 0$ on EF also. Finally, in order that $f = f_1 + f_2$ on LM, one has to set $f_1(x') = f(\xi_3)$, where $\xi_3 \in LM$ is the point of LM with abscissa x'. It is easily seen that the constructed functions $f_1(x)$ and $f_2(y)$ are the desired ones.

It is easy to devise an analogous construction for the function given on a more complicated tree (Figure 11). In general, we have the following type of theorem.

Every finite tree, whose branch points are of index not greater than 3, can be mapped homeomorphically onto a flat segment-like complex K such that every continuous function $f(\xi)$ is representable on K in the form $f(\xi) = f_1(x) + f_2(y)$, where x and y are the coordinates of the point $\xi \in K$.***

Theorem 3 asserts that an analogous result holds in the three-dimensional space for any tree whose points have a branching index not greater than three. The proof is very involved, but can be reduced in essence to the considerations given above, and to the transition to the infinite tree from finite trees.

Theorem 1 is a direct consequence of Theorems 2 and 3. Taking the risk of possibly confusing the reader, who could derive the proof himself, we nevertheless present a simple argument.

From Theorem 2 it follows that one can express the function $f(x_1, x_2, x_3)$ as the sum of three functions $h_i(\xi_i, x_3)$ $(i = 1, 2, 3)$ from the product of the tree $(\xi_i \in \Xi)$, none of whose points have a branching index greater than 3, by the segment $(x \in E^1)$: $(\xi_i, x_3) \in \Xi \times E^1$. Theorem 3 asserts that the function $h(\xi)$ on such a tree can be expressed as the sum of three continuous

* A tree with a finite number of points.

** The reader can easily construct the proof of this theorem after he reads §3-7. Whether it is possible to give an analogous representation for an infinite tree, is not known.

functions $h_j(x_j)$ $(j = 1, 2, 3)$ of the coordinates x_j of some realization $x_j(\xi)$ of the tree Ξ in the three-dimensional space. These functions $h_j(x_j)$ depend continuously on the decomposed function $h(\xi)$ (in the sense of uniform convergence) if the function h belongs to the same family F of equi-continuous functions on the tree Ξ for which the realization is constructed. The functions $h_i(\xi_i, x_3)$ that are obtained from Theorem 2 can be considered to be such a family of functions $h_i(\xi)$ on the tree Ξ, which depend continuously on the parameter $x_3 \in E^1$, and they are, therefore, equi-continuous. Applying Theorem 3, we find a realization of Ξ in the form $X \subset E^3$.

In the decomposition $f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i(\xi_i, x_3)$, $\xi_i = \varphi_i(x_1, x_2)$ is a point of the tree Ξ and depends continuously on x_1 and x_2 (Theorem 2). Hence, after the realization of Ξ in the form X, every coordinate $x \in X$ becomes a real, continuous function of x_1 and x_2. If $\xi_i = \varphi_i(x_1, x_2)$ and the jth coordinate of the point x that is realized by ξ_i is $\varphi_{ij}(x_1, x_2)$, then, in view of Theorem 3, the decomposition of $h_i(\xi_i, x_3)$, as a function of $h_{i x_0}(\xi_i)$, into the sum $\sum_{j=1}^{3} h_{ij x_0}(x_j(\xi_i))$ can be written in the form

$$h_i[\xi_i(x_1, x_2), x_3] = \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3].$$

Therefore,

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1, x_2), x_3],$$

which is the assertion of Theorem 1.

About two months after the completion of our work [1], A.N. Kolmogorov [2] strengthened the Theorem 1 by showing that every continuous function on the three-dimensional cube is representable in the form

$$f(x_1, x_2, x_3) = \sum_{i=1}^{7} h_i[\varphi_{i1}(x_1) + \varphi_{i2}(x_2) + \varphi_{i3}(x_3)],$$

where the functions h_i and φ are continuous; the functions φ_{ik} are, however, selected once for all independently of f. From this result of A.N. Kolmogorov it follows that the three-dimensional cube can be imbedded in a seven-dimensional space so that any continuous function on the cube will be expressible as the sum of continuous functions of (seven-dimensional) coordinates. According to the work [2], an analogous representation for a square can be realized in a five-dimensional space. From this it follows directly that in a five-dimensional space we can place our tree Ξ, once for all, so that any function continuous on it is expressible as the sum of continuous functions of the coordinates (while in our Theorem 3 the representation in the three-dimensional space depended on the family F). But by modifying the methods of the note [2], one can obtain a representation of the tree Ξ which is valid for all continuous functions f in the three-dimensional space also.

In the constructions of the first and second parts of the present work, use is being made of the tree of the components of the level sets, which was introduced by A.S. Kronrod. The essential information about this tree can be found in the Appendix. The Appendix and each of the two parts of this work are independent of each other.

I take this opportunity to thank my teachers A.G. Vituškin and A.N. Kolmogorov for their constant attention, counsel and help. In particular, I am indebted to A.N. Kolmogorov for the final formulation of the fundamental "inductive lemma" of the second part.

PART I

Proof of Theorem 2

Here we shall prove Theorem 2. The fundamental lemma of the work [2] and Lemma 2 are proved in such a formulation that the tree Ξ, under consideration in Theorem 3, consists of points whose branching index does not exceed 3.

The following notations will be used:

R^2 is the plane of the (x,y) points; E^2 is the closed unit square in this plane, i.e., the set of points (x,y) with $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

The metric in the plane is defined as the distance

$$\rho((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

$U_d(A)$ denotes a d-neighborhood of the set A, i.e. the set of all points in the plane whose distance from the set A is less than d $(d > 0)$.

$\bar{A}$ is the closure of A.

A polygon is a closed broken line that does not intersect itself. An open polygon Q is the part of the plane lying inside a polygon, while a closed polygon $\bar{Q}$ is the closure of the open polygon.

An open polygonal band is the part of the plane bounded by two nonintersecting polygons, one of which lies inside the other (is separated by the other from infinity). A closed polygonal band is the closure of an open one.

The set of the level c of a function $u(x,y)$ is the set of points (x,y) such that $u(x,y) = c$.

A list of the topological terms used in this work is given at the end of the Appendix.

§1. Fundamental lemma

Suppose that we are given a finite number of nonintersecting regions g_m in a plane, and that over each region there is a hill u_m. The set of hills form a "mountain country" G. Suppose that we are given not only one mountain country G (Figure 2) but an infinite sequence Γ of such mountain countries,

$$G_1, G_2, \ldots, G_k, \ldots,$$

where the "country of rank k" G_k consists of some finite number m_k of hills u_{km} of rank k $(m = 1, \ldots, m_k)$ over the regions g^r_{km}; no two regions of a given mountain country intersect each other (Figure 2). For large k, the country G_k has more hills, but their bases, the regions g^r_{km}, are smaller.

Finally, let us suppose (and this is not shown in Figure 2) that we are given three such sequences of countries Γ^r $(r = 1, 2, 3)$, namely, three systems Γ^r. Each of them consists of mountain countries G^r_k $(k = 1, 2, \ldots)$, and each mountain country G^r_k consists of hills u^r_{km} $(m = 1, \ldots, m_k)$.

In the fundamental lemma there are constructed three such systems of hills u^r_{km} satisfying a number of requirements. For example, every hill u^r_{km} is constructed in such a way that over every region $g^r_{k'm'}$ $(k' > k)$ it possesses a horizontal plane (requirement 5).

Fundamental lemma. *It is possible to define on the plane* R^2 *a system of real functions* $u^r_{km}(x,y)$, *with indices lying within the limits*

$$1 \leqslant r \leqslant 3, \quad 1 \leqslant k < \infty, \quad 1 \leqslant m \leqslant m_k,$$

and having the following properties:

1) $0 \leqslant u^r_{km} \leqslant 1$.

2) $u^r_{km} \neq 0$ *just on the region* g^r_{km} *whose diameter is less than* $d_k > 0$; $d_k \to 0$ *as* $k \to \infty$; $u^r_{km} = 1$ *on the set* $g^r_{k+1\, m^*}$ *only.*

3) *Two sets* g^r_{km} *and* $g^r_{km'}$ *with the same indices* r *and* k, *but* $m \neq m'$, *do not intersect.*

4) *For any given* k, *and for every point of the square* E^2, *it is true that*

$$0 < c \leqslant \sum_{r=1}^{3} \sum_{m=1}^{m_k} u^r_{km} \leqslant C,$$

where c *and* C *are constants independent of* k.

5) *The function* u^r_{km} *is constant on each set* $g^r_{k'm'}$ *with the same index* r *when* $k' > k$ *but* m *and* m' *arbitrary.*

6) *The boundary of each level set of the function* u^r_{km} *is connected and divides the plane* R^2 *into three parts at most.*

7) *For every* r, $g^r_{11} \supset E^2$.

The functions u^r_{km} and the sets g^r_{km} with the same index $r = r_0$ will be called functions and sets of the one system r_0, while those with the same index k (and arbitrary r and m) will be said to be functions and sets of the same rank. The index m will be called *number*. Obviously, for any N the totality of functions (sets) of rank not higher than N in each system will be finite.

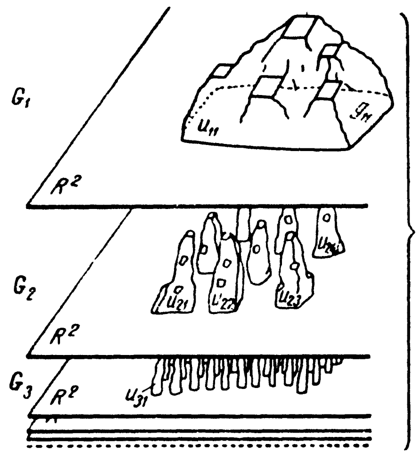


Figure 2. System of mountain countries. All the horizontal planes R^2 are actually in the same plane.

It is known that for every $\varepsilon > 0$, the bounded region $E \supset E^2$ of the plane R^2 can be enclosed (covered) by means of closed squares $P_{\varepsilon m}$, whose sides are parallel to the coordinate axes, in such a way that the set of squares can be divided into three systems $p^r_{\varepsilon m}$, $1 \leqslant r \leqslant 3$, whereby the distance between any two squares of one system will be greater than $\varepsilon / 2$ (Lebesgue covering, Figure 3). These squares are the cells of the regions g^r_{km}.

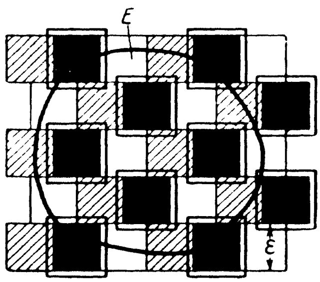


Figure 3. Lebesgue covering. The squares of one system are lined, those of another system are black, those of the third one are white. The functions $Q^r_{\varepsilon_{km}}$ are constructed for the black squares $P^r_{\varepsilon_{km}}$.

All the successive constructions for each r are done independently. During each of the constructions of the functions u^r_{km}, r is kept fixed.

The sets g^r_{km} ($m = 1, \ldots, m_k$) are obtained from the squares $P^r_{\varepsilon_{km}}$, where $\varepsilon_k > 0$. The selection of the number ε_k will be described later. The regions g^r_{km} will be obtained by means of a "dilatation" of the $P^r_{\varepsilon_{km}}$ in such a way that $P^r_{\varepsilon_{km}} \subset g^r_{km} \subseteq Q^r_{\varepsilon_{km}}$, where $Q^r_{\varepsilon_{km}}$ is the closure of the square which is an $(\varepsilon_k/6)$-neighborhood of $P^r_{\varepsilon_{km}}$: $\overline{U}_{\varepsilon_{k/6}}(P^r_{\varepsilon_{km}}) = Q^r_{\varepsilon_{km}}$ (see Figure 3).

It is obvious that if $m_1 \neq m_2$, $\rho(Q^r_{\varepsilon_{km_1}}, Q^r_{\varepsilon_{km_2}}) > \varepsilon_{k/6}$. Therefore $\rho(\overline{g}^r_{km_1}, \overline{g}^r_{km_2}) \geqslant \varepsilon_{k/6}$.

This means that by this construction the requirement 3) of the fundamental lemma will be satisfied.

In order to fulfil the requirement 2), it is obviously necessary that $\varepsilon_k \to 0$ as $k \to \infty$. It will become obvious that this condition will be fulfilled by the construction given below.

This construction is divided into several stages. Everything that is constructed at the nth stage will carry the superscript n together with that of the system r.

In general, all notations are constructed so that $A^{rn}_{\varepsilon_{km}}$ should be read as follows: the object A is constructed for the function u (or the set g) of the system r of rank k and number m, i.e. for u^r_{km} (g^r_{km}) at the nth stage. The letters have the following designations:

P is the square cell.

Q is an approximation to g from within.

$\hat{Q}$ is an approximation to g from without.

${}^xO$ is an approximation to the set of the level $u = x$ $(0 < x < 1)$ and to the boundary of the set of the level $u = x$ when $x = 0$ and $x = 1$.

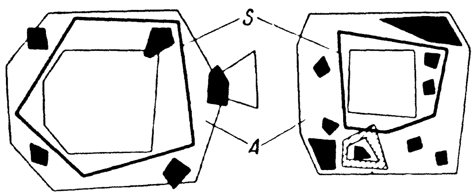
${}^x\Theta$ is an approximation to the boundary of the set of the level $u = x$ $(0 < x < 1)$.

For example, ${}^{x_i}\Theta^{rn}_{\varepsilon_{km}}$ denotes the approximation to the boundary of the set of the level $u^r_{km} = x_i$ constructed at the nth stage.

We start the construction of the g^r_{km} at the kth stage, but at the nth stage $(n \geqslant k)$ we construct the $(n-k+1)$st approximation to g^r_{km} from within and from without: $Q^{rn}_{\varepsilon_{km}} \subseteq g^r_{km} \subset \hat{Q}^{rn}_{\varepsilon_{km}}$. Hereby $Q^{rn+1}_{\varepsilon_{km}} \supseteq Q^{rn}_{\varepsilon_{km}}$ and g^r_{km} is determined as $\bigcup\limits_{n=k}^{\infty} Q^{rn}_{\varepsilon_{km}}$, i.e. as the sum of the dilated approximations from within.

The functions u^r_{km} are constructed with the aid of their level sets. The construction is begun at the kth stage where one constructs the first approximation ${}^0O^{rk}_{\varepsilon_{km}} = \hat{Q}^{rk}_{\varepsilon_{km}} \setminus Q^{rk}_{\varepsilon_{km}}$ to the set of the zero level. At the next stage one constructs the first approximations to the sets of the levels 1/2 and 1, and to certain other levels, and the second approximations to the zero level. At each stage there appear first approximations to new levels, and one makes successive approximations to the earlier used levels. Each approximation is a closed polygonal band imbedded in the preceding approximation, while the level set itself is the intersection of all its approximations. The values of the function u on each of such level sets are selected so that u^r_{km} is continuous, positive on g^r_{km}, larger than 1/2 on $P^r_{\varepsilon_{km}}$ but does not exceed 1 anywhere. The requirements 1) and 4) of the fundamental lemma will thus be satisfied.

We shall make use of an elementary geometric lemma whose proof will be omitted. It is sufficient to examine Figure 4 to convince oneself of the truth of this lemma.



Case 1) Case 2)

Figure 4. The polygons Q_m are black.
The band B is lined.

Geometric lemma. *Let A be a closed polygonal band whose width (i.e. the smallest distance between the boundaries of the polygons) is greater than a positive number d. Let the Q_m $(m = 1,\dots,M)$ be closed nonintersecting polygons.*

1) *If the diameter of each of the polygons Q_m does not exceed d, then it is possible to construct a polygon S which is strictly inside the band A, separates the boundary of the band A, and does not intersect the polygons Q_m $(m = 1,\dots,M)$.*

2) *If another closed polygonal band B lies strictly inside the band A, and if the polygons Q_m do not intersect the boundaries of A and B, then the polygon S, which separates the boundaries of A and does not intersect Q_m, can be drawn strictly within the band A so that its intersection with B will be an interval (segment).*

We now begin the construction at the first stage.

In order to fulfil the requirement 7) of the fundamental lemma, we set

$\varepsilon_1 = 1$, $m_1 = 1$, $P^r_{11} = E^2$. We construct the squares (Figure 5)

$$Q^r_{11} = \hat{Q}^{r1}_{11} = \overline{U_{\frac{1}{6}}(P^r_{11})} \text{ and } Q^{r1}_{11} = U_{\frac{1}{12}}(P^r_{11}).$$

This is the first approximation to g^r_{11}, for we see that $Q^{r1}_{11} \subseteq g^r_{11} \subseteq \hat{Q}^{r1}_{11}$. $\hat{Q}^{r1}_{11} \setminus Q^{r1}_{11} = {}^0O^{r1}_{11}$ is called the first approximation to the boundary g^r_{11}. This is a closed polygonal band of width 1/12.

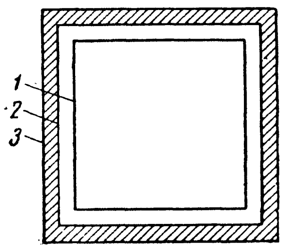


Figure 5. *1* is the boundary of P^r_{11}, *2* is the boundary of Q^{r1}_{11}, *3* is the boundary of $\hat{Q}^{r1}_{11}$. The shaded band ${}^0O^{r1}_{11}$ is the first approximation to the boundary of the region g^r_{11}.

If $\varepsilon_2 < (1/12)\cdot(3/4)$, then the squares $Q^r_{\varepsilon_2 m}$ ($m = 1, \ldots, m_2$) can be taken for the Q_m in the geometric lemma,* while the first approximation to the boundary of g^r_{11} plays the role of A.

With this selection of ε_2, we start the second stage (Figure 6). For this ε_2 we construct the squares $P^r_{\varepsilon_2 m}$, $Q^r_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_2 m} = \overline{U_{\varepsilon_2/6}(P^r_{\varepsilon_2 m})}$; $Q^{r2}_{\varepsilon_2 m} = U_{\varepsilon_{2/12}}(P^r_{\varepsilon_2 m})$ ($m = 1, \ldots, m_2$). The $Q^{r2}_{\varepsilon_2 m}$ are the first approximations to the regions g^r_{2m}, while the ${}^0O^{r2}_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_2 m} \setminus Q^{r2}_{\varepsilon_2 m}$ are the first approximations to their boundaries.

It will be convenient to perform the construction so that the boundaries of the regions g^r_{km} and $g^r_{k'm'}$ do not intersect. It can happen that this requirement is not fulfilled for the first approximation: the band ${}^0O^{r1}_{11}$ may intersect the squares $Q^r_{\varepsilon_2 m}$. However, on the basis of the geometric lemma one can draw a polygon within this band which separates Q^{r1}_{11} from infinity and winds among the squares $Q^r_{\varepsilon_2 m}$ without touching them. This polygon, naturally, can be enclosed in the closed polygonal band ${}^0O^{r2}_{11}$ which will be the second approximation to the boundary of g^r_{11} or to the boundary of the set of the level $u^r_{11} = 0$. (This explains the use of the left 0 superscript.) The band ${}^0O^{r2}_{11}$ determines the second approximation Q^{r2}_{11} to g^r_{11} and can be represented in the form $\hat{Q}^{r2}_{11} \setminus Q^{r2}_{11}$.

At the second stage we construct also the first approximations to certain other level sets of the function u^r_{11}. It is easy to see that, since $\varepsilon_2 < (3/4)\cdot(1/12)$, one can find a square $Q^r_{\varepsilon_2 m^*}$ which will lie entirely within P^r_{11}. It is the first approximation to the set of the level 1 for the function

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* The construction of the squares is described after the formulation of the fundamental lemma (see Figure 3). For the region E, which occurs there, one should take Q^r_{11}.

u^r_{11}, while the band ${}^1O^{r2}_{11} = \hat{Q}^{r2}_{\varepsilon_2 m^*} \setminus Q^2_{\varepsilon_2 m^*}$ is the first approximation to the boundary of this set.

Next, in order to satisfy the requirement 4), we construct the set of the level 1/2. The boundaries of P^r_{11} and Q^{r1}_{11} are at a distance of 1/12 from each other, while $\varepsilon_2 < (3/4)\cdot(1/12)$. Therefore, applying the geometric lemma to the band between P^r_{11} and Q^{r1}_{11}, we construct a polygon, and then a closed band ${}^{1/2}O^{r2}_{11}$, which winds among the squares $\hat{Q}^{r2}_{\varepsilon_2 m}$ $(m = 1, \dots, m_2)$ without touching them, lies within Q^{r1}_{11}, and separates P^r_{11} from infinity. The band ${}^{1/2}Q^{r2}_{11}$ becomes the first approximation to the set of the level 1/2 for the function u^r_{11}. The successive approximations ${}^{1/2}\,Q^{rn}_{11}$ $(n > 2)$ are constructed within this band.

Finally, one constructs at the second stage the first approximations to the sets of the levels of the function u^r_{11} that contain g^r_{2m}, and one determines the values u^r_{11} on these sets.

First of all we discard forever those squares $Q^r_{\varepsilon_2 m}$ which were found to lie outside Q^{r2}_{11} (and, hence, outside $\hat{Q}^{r2}_{11}$). The remaining squares $Q^r_{\varepsilon_2 m}$ $(m \in M^{r2}_{11})$ (excluding $Q^r_{\varepsilon_2 m^*}$) *lie in the ring-shaped regions* into which Q^{r2}_{11} is divided by the *finished* bands ${}^0O^{r2}_{11}$, ${}^{1/2}O^{r2}_{11}$ and ${}^1O^{r2}_{11}$. Each ring-shaped region is an open polygonal band which separates $Q^r_{\varepsilon_2 m^*}$, and everything that lies within it, from infinity.

Let us consider any one square $Q^r_{\varepsilon_2 m_0}$ $(m_0 \in M^{r2}_{11},\ m \neq m^*)$. We take the closure of the polygonal band in which the given square lies, for the band A of the geometric lemma; the remaining squares $Q^r_{\varepsilon_2 m}$ $(m \neq m_0)$ we take for the Q_m, and the band ${}^0Q^{r2}_{\varepsilon_2 m} = \hat{Q}^{r2}_{\varepsilon_1 m_0} \setminus Q^{r2}_{\varepsilon_2 m_0}$ for the band B. In accordance with this lemma, we now draw the polygon S, which intersects ${}^0O^{r2}_{\varepsilon_2 m_0}$ in an interval, separates $\overline{Q^r_{\varepsilon_2 m^*}}$ from infinity, lies inside the open polygonal band between the finished bands, and does not touch the squares $\overline{Q^r_{\varepsilon_2 m}}$ $(m \neq m_0)$. This polygon S can be enclosed in a closed polygonal band d, which has the same properties, in such a way that $d \cup Q^r_{\varepsilon_2 m_0}$ is also a closed polygonal band (Figure 6). It is ${}^{x_0}O^{r2}_{11}$, the first approximation to the level set, for the function u^r_{11}, that contains $g^r_{2m_0}$. The value x_0 of the function u^r_{11} on this level set is determined below.

Adding the band ${}^{x_0}O^{r2}_{11}$ to the finished ones, we choose from the M^{r2}_{11} a new $m \neq m_0$, $m \neq m^*$, and construct by the same method an ${}^{x_1}O^{42}_{11}$, and so on, until the set M^{r2}_{11} is exhausted and every square $Q^r_{\varepsilon_2 m}$ $(m \in M^{r2}_{11})$ is enclosed in the first approximation to some level set of the function u^r_{11}. These approximations are polygonal closed nonintersecting bands. The sets ${}^{x_i}\Theta^{r2}_{11} = {}^{x_i}O^{r2}_{11} \setminus O^{r2}_{\varepsilon_2 m_i}$ are called Θ-type closed bands. Each of them divides the plane into three parts: the $Q^{r2}_{\varepsilon_{2m_i}}$, the part that contains $^1O^{r2}_{11}$, and the part that contains $^0O^{r2}_{11}$ (Figure 6). They are the first approximations to the boundaries of the level sets of u^r_{11} containing $g^r_{2m_i}$ $(m_i \in M^{12}_{11}\,;\ m_i \neq m^*)$.

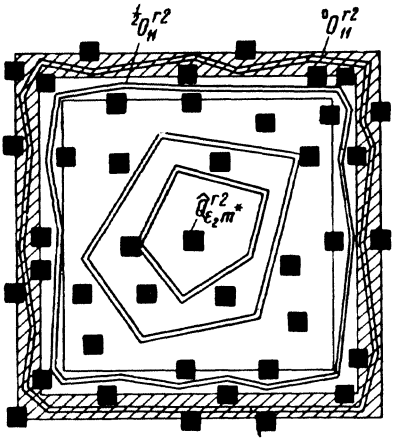


Figure 6. The thin lines are first stage constructions. The black squares are the $Q^r_{\varepsilon_{2m}}$. There should be many more of them but then one would not be able to see anything on the figure. The black square at the center is $Q^r_{\varepsilon_{2m^*}}$. Only a few of the bands containing the squares are shown; the second stage is not completed.

Finally, let us determine the values x_i.

Between the boundary of the set g^r_{11} and $^{1/2}O^{r2}_{11}$, the function u^r_{11} will increase from 0 to 1/2, while between $^{1/2}O^{r2}_{11}$ and $^1O^{r2}_{11}$, from 1/2 to 1. The bands $^{x_i}O^{r2}_{11}$ are divided by $^{1/2}O^{r2}_{11}$ into two classes: p_1 outer bands lying outside $^{1/2}O^{r2}_{11}$, and p_2 inner ones. Let us reorder them by means of an index $j = j(i)$ in the order determined by their separation from infinity: the outer ones from 1 to p_1, the inner ones from $p_1 + 1$ to $p_1 + p_2$. Let us spread out the increase of u from 0 to 1/2 uniformly among the outer bands, by letting the jth band be an approximation to the set of the level $u^r_{11} = j/2(p_1 + 1)$. For the inner bands of uniform increase from 1/2 to 1, we let the jth band be the approximation to the set of the level $1/2 + (j - p_1)/2(p_2 + 1)$.

Thus we have obtained the following objects at the second stage:

1) The first approximations $Q^{r2}_{\varepsilon_{2m}}$ and $^0O^{r2}_{\varepsilon_{2m}}$ to the sets g^r_{2m} and their boundaries.

2) The second approximations Q^{r2}_{11} and $^0O^{r2}_{11}$ to the g^r_{11} and its boundary.

3) The first approximations to the set of the level 1 of the function u^r_{11} and to its boundary, to the set of the level 1/2, and to the sets of the levels of u^r_{11} on which the $g^r_{2m_i}$ $(m_i \neq m^*;\ m \in M^{r2}_{11})$ lie, and also to the boundaries of these sets, $\hat{Q}^{r2}_{\varepsilon_{2m^*}}$, $^1O^{r2}_{11}$, $^{1/2}O^{r2}_{11}$, $^{x_i}O^{r2}_{11}$, $^{x_i}\Theta^{r2}_{11}$.

4) The values x_i of the function u^r_{11} on the $g^r_{2m_i}$, and on the level sets that contain them (not yet constructed).

The approximations to the open sets are open polygons containing the preceding approximations, while the approximations to the closed sets are closed polygons, polygonal bands, and Θ-type bands contained within the preceding approximations.

We note that the construction at the second stage of the functions and sets of rank 2, is exactly the same procedure (if one disregards the scale ε_2) as that used at the first stage for the construction of the functions and sets of rank 1.

In general, after the nth stage we will have:

1) the first stage of the construction of the functions and sets of the nth rank, the second stage of the construction of the functions and sets of the $(n-1)$st rank, and so on up to the $(n-1)$st stage of the construction of u^r_{2m} and g^r_{2m};*

2) the nth approximations Q^{rn}_{11} and ${}^{o}O^{rn}_{11}$ to the set g^r_{11} and its boundary, respectively;

3) the $(n-1)$st approximation to the level sets of the function u^r_{11}, which we began to construct at the second stage, and the $(n-2)$nd approximations to those level sets which we began to construct at the third stage, and so on up to the first approximations ${}^{x_i}O^{rn}_{11}$ and ${}^{x_i}\Theta^{rn}_{11}$ to the level sets of u^r_{11} that contain the $g^r_{nm_i}$, and to the boundaries of these level sets. Here $m_i \in M^{rn}_{11}$, i.e. m_i runs through those values from 1 to m_n for which the corresponding squares $Q^r_{\varepsilon_n m_i}$ do not lie within $Q^{rn}_{\varepsilon_k m'}$ $(1 < k < n;\ m' \leqslant m_k)$, but lie inside Q^{rn}_{11};

4) the values x_i of the function u^r_{11} on g^r_{nm} $(m \in M^{rn}_{11})$.

We have the following results.

1°. The approximations to the open sets are open polygons whose boundaries do not intersect each other (nor, in particular, the small squares $Q^r_{\varepsilon_n m} = \hat{Q}^{rn}_{\varepsilon_n m}$). These approximations contain the preceding ones.

2°. The approximations to the closed sets are closed polygons, closed polygonal, or polygonal Θ-type bands enclosed in the preceding approximations. The polygons that are the boundaries of these approximations do not intersect the other polygons constructed at the nth stage (nor, in particular, the boundaries of the small squares $Q^r_{\varepsilon_n m}$).

3°. Each one of the bands ${}^{x_i}O^{rn}_{11}$, and each of the ${}^{x_i}\Theta^{rn}_{11}$ $(m_i \in M^{rn}_{11})$ contained in it, separates $\hat{Q}^{rn}_{\varepsilon_2 m^*}$ from infinity, while ${}^{x_i}\Theta^{rn}_{11}$, besides that, separates from the rest of $Q^{rn}_{\varepsilon_n m_i} \subset {}^{x_i}O^{rn}_{11}$ the first approximation to the set

* We call attention to the fact that the notation always reflects the number of the stage at which an object is constructed and not the number of the approximation. For example, $Q^{rn}_{\varepsilon_n m}$ is the first approximation to $g^r_{\varepsilon_n m}$.

$g^r_{nm_i}$ which lies on the set of the level $u^r_{11} = x_i$.

4°. The values of u^r_{11} are uniformly distributed on g^r_{nm} ($m \in M^{rn}_{11}$).

The last phrase has the following meaning by definition.

Let the bands A and B be constructed at the nth stage of the approximation to the set of the levels a and b of the function u^r_{11}, where a and b are determined up to the nth stage. Suppose that at the $(n-1)$st stage there was no band (of the approximation to the set of the level u^r_{11}) between A and B, but at the nth stage such bands C_i ($i = 1, \ldots, p$) were constructed (the numbering of the C_i is from A to B). If the value x_i of the function u^r_{11} on the level set for which C_i is the first approximation is equal to $i(b-a)/(p+1)$ then the values of u^r_{11} on g^r_{nm} are said to be distributed uniformly between A and B. The condition 4) requires that the values of u^r_{11} on g^r_{nm} be so constructed between any two bands A and B of the indicated type.

The $(n+1)$st stage begins with the selection of an ε_{n+1}. Since any two of the polygons that bound the nth stage approximations to all level sets of all the functions u^k_{rm} $(k \leqslant n)$ and to their boundaries do not intersect (provided they are not identical), there exists a positive number d such that the distance between any two distinct polygons is greater than d. We choose ε_{n+1} so that $\varepsilon_{n+1} < 3d/4$. This ε_{n+1} permits us to carry out the first stage of the construction of the sets $g^r_{n+1\,m}$ and of the functions $u^r_{n+1\,m}$, the second stage of the construction of g^r_{nm} and u^r_{nm}, and so on up to the nth stage of the construction of g^r_{2m} and u^r_{2m}.

Since we now assume that we have gone through the stages of rank less than $n+1$ for u^r_{11}, and since they are entirely analogous for the remaining g^r_{km} and u^r_{km} $(k \leqslant n)$, we consider only, as an example, the first stage of the construction of the sets $g^r_{n+1\,m}$ and of the functions $u^r_{n+1\,m}$.

For ε_{n+1} we construct a Lebesgue covering with the squares $P_{\varepsilon_{n+1}m}$ of the nth approximations $\hat{Q}^{rn}_{11}$ to g^r_{11} from without. We divide this covering into three systems $P^r_{\varepsilon_{n+1}m}$, and construct with them the first approximations to g^r_{nm} from within and from without,

$$Q^{r\,n+1}_{\varepsilon_{n+1}m} = U_{\frac{\varepsilon_{n+1}}{12}}(P^r_{\varepsilon_{n+1}m}), \quad Q^r_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m} = \overline{U_{\frac{\varepsilon_{n+1}}{6}}(P^r_{\varepsilon_{n+1}m})}$$

$$(m = 1, \ldots, m_{n+1})$$

and the first approximations to the boundaries of g^r_{nm}

$${}^0O^{r\,n+1}_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m} \setminus Q^{r\,n+1}_{\varepsilon_{n+1}m} \quad (m = 1, \ldots, m_{n+1}).$$

(The squares $Q^r_{\varepsilon_{n+1}m}$ will be called the small squares.)

Since $\varepsilon_{n+1} < 3d/4$, one can now proceed with the second stage of the construction of g^r_{nm} and u^r_{nm}, and so on to the nth stage of the construction g^r_{2m} and u^r_{2m}.

Suppose all this has been done. Then one has to carry out the $(n+1)$st stage of the construction of g^r_{11} and u^r_{11}.

Let us consider any closed band ${}^xO^{rn}_{11}$ which is an approximation to the set of the level x of the function u^r_{11}. If $x = 0$, or $x = 1/2$, then ${}^xO^{rn}_{11}$ will not intersect the sets $\hat{Q}^{rn}_{km}$ $(k \leqslant n)$. It can intersect the squares $Q^r_{\varepsilon_{n+1}m} = \hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m}$, but their diameters are less than d, which is less than or equal to the width of the band. Therefore, applying the geometric lemma, and expanding the polygon S up to the closed polygonal band which winds within the band ${}^xO^{rn}_{11}$ without touching the small squares, we obtain the bands ${}^0O^{r\,n+1}_{11}$ and ${}^{1/2}O^{r\,n+1}_{11}$ that satisfy all the requirements 1° to 4°.

If $x = 1$, then ${}^1O^{r\,n+1}_{11}$ will be ${}^1O^{r\,n+1}_{\varepsilon_{2m^*}}$, a band that already has been constructed, since we assume that the nth stage of the construction of the functions $u^r_{2m^*}$ has been completed.

If $x \neq 0$, $1/2$, or 1, then the band ${}^{x_i}O^{rn}_{11}$ contains the approximation $\hat{Q}^{rn}_{\varepsilon_{km_i}}$ to $g^r_{km_i}$ $(k \leqslant n)$, which was constructed at the nth stage, and this band contains, therefore, also the band ${}^0O^{r\,n+1}_{\varepsilon_{km_i}}$ that has been constructed at the $(n+1)$st stage. Since this band, which contains $O^{rn}_{\varepsilon_{km_i}}$, and is contained in $\hat{Q}^{rn}_{\varepsilon_{km_i}}$, does not intersect the small squares, one can choose it for the band B in the geometric lemma, while for the band A of that lemma, we can take ${}^{x_i}O^{rn}_{11}$. Applying the lemma, we obtain a polygon S which 1) intersects the band ${}^0O^{r\,n+1}_{\varepsilon_{km_i}}$ in an interval, 2) separates ${}^1O^{r\,n+1}_{11}$ from ${}^0O^{r\,n+1}_{11}$, 3) lies inside ${}^{x_i}O^{rn}_{11}$, and 4) winds among the small squares without touching them. Dilating S to the closed polygonal band d, which has the properties 2), 3), and 4) and which is such that ${}^{x_i}O^{r\,n+1}_{11} = d \cup \hat{Q}^{r\,n+1}_{\varepsilon_{km_i}}$ is also a closed polygonal band (that this is possible is obvious), we obtain the following approximation ${}^{x_i}O^{r\,n+1}_{11}$ to the set of the level $u^r_{11} = x_i$.

${}^{x_i}\Theta^{r\,n+1}_{11} = {}^{x_i}O^{r\,n+1}_{11} \setminus Q^{r\,n+1}_{\varepsilon_{km_i}}$ is the next approximation to the boundary of this level set.

Having completed the indicated operation for all the bands ${}^{x_i}O^{rn}_{11}$, we will have the set of all closed polygonal nonintersecting bands ${}^xO^{r\,n+1}_{11}$ that separate ${}^1O^{r\,n+1}_{11}$ from infinity. These bands will be referred to as finished bands.

Let us begin to construct first approximations to the level sets of the function u^r_{11} that contain the sets $g^r_{n+1\,m}$ of rank $n+1$.

The boundaries of the finished bands ${}^{x_i}O^{r\,n+1}_{11}$ do not intersect the small squares. Let us consider the numbers m that correspond to those small squares that lie in $Q^{r\,n+1}_{11}$, and do not lie in any of the finished bands. The set of all such m, we denote by $M^{r\,n+1}_{11}$. The small squares $\hat{Q}^{r\,n+1}_{\varepsilon_{n+1}m}$ $(m \in M^{r\,n+1}_{11})$ must be included in the first approximations to the level sets. The finished bands divide the $Q^{r\,n+1}_{11}$ into open polygonal bands which contain the small squares. In each of these bands we proceed exactly as it was described in the performance at the second stage. The only difference is that we now have more finished bands. As a result, we obtain the bands ${}^{x_i}O^{r\,n+1}_{11}$ and the Θ-type bands ${}^{x_i}\Theta^{r\,n+1}_{11}$ which are approximations to the level sets and their boundaries. The values x_i in each open band between two finished bands are distributed uniformly.

In this manner one can accomplish the construction by building at each stage objects that have the properties 1°, 2°, 3°, 4°.

Suppose that all stages have been completed.

We define g^r_{km} as $\bigcup_{i=k} Q^{ri}_{km}$.

The level sets of the function u^r_{km} which contain the sets $g^r_{k'm'}$ $(k' > k)$ are defined as the intersections of the corresponding polygonal bands, the approximations. The values of the function on these levels are determined at the k'th stage.

On all regions g^r_{km} the functions u^r_{km} are extended by continuation. Below it is proved that this can be done, and that the obtained functions will satisfy all the requirements of the fundamental lemma.

It is obvious that $\varepsilon_k \to 0$ when $k \to \infty$. Recalling how the squares P were constructed, we see that $\bigcup_{\substack{ij \\ P^r_{k_i m_j} \subset g^r_{km}\ (k_i > k)}} P^r_{k_i m_j}$ is an everywhere dense set on g^r_{km}. Because of this, the sum Σ^r_{km} of all level sets on which we determined u^r_{km} is everywhere dense in g^r_{km}. We shall show that the function u^r_{km} is uniformly continuous on the set Σ^r_{km}.

Without restricting the argument, we will set $k = m = 1$, and will give the proof only for $u^r_{11} = u$.

Let $\varepsilon > 0$ be given. At each $(n+1)$st stage one can find between any two bands ${}^{x}O^{r\,n+1}_{11}$, ${}^{y}O^{r\,n+1}_{11}$ at least one square $\hat{Q}^r_{\varepsilon_{n+1}m}$, if the construction of the levels $u = x$ and $u = y$ began before the $(n+1)$st stage. Indeed, the width of the open band O^n between ${}^{x}O^{rn}_{11}$ and ${}^{y}O^{rn}_{11}$ is greater than d, while the squares $P^r_{\varepsilon_{n+1}}$ have diameters less than d and enter into

Lebesgue covering in such a way that one of them $P^r_{\varepsilon_{n+1}\overline{m}}$ has points in O^n. This square, and with it $\hat{Q}^{r\,n+1}_{\varepsilon_{n+1}\overline{m}}$ will, obviously, fall into the open band O^{n+1} between ${}^xO^{r\,n+1}_{11}$ and ${}^yO^{r\,n+1}_{11}$. But at the $(n+1)$st stage $(n > 1)$ the values between the newly constructed band were distributed uniformly. Therefore, the largest interval between the values of u on two level sets, whose approximations are neighboring bands of the nth stage, will decrease by two at each stage. Hence, there exists a stage k such that if ${}^xO^{rk}_{11}$ and ${}^yO^{rk}_{11}$ are neighboring bands, then $|x - y| < \varepsilon/2$.

Let us select $\delta = \varepsilon_{k+1}$. Suppose that $\rho(a,b) < \delta$. Then the points a and b are separated by one band ${}^zO^{rk}_{11}$ only, since the distance between the polygons that bound the bands is greater than $\varepsilon_{k+1} = \delta$. Hence, there exists a band ${}^zO^{rk}_{11}$ which is not separated from a and b by any other band. But it is obvious that at such points the function u differs from z by less than $\varepsilon/2$ (the rank k is chosen in this way). Therefore, $|u(a) - u(b)| < \varepsilon$, and the function u is thus uniformly continuous on the everywhere dense set of the compact $\overline{g}$.

This function can be extended (and in a unique manner) over the set $\overline{g}$.

We set $u = 0$ outside of g. Such a continuation of the functions u^r_{km} will satisfy the requirements 1) to 7) of the fundamental lemma.

Indeed, the fulfillment of the requirements 1), 2), 3), and 7) is obvious.

The condition 4) is satisfied with the constants $c = 1/2$ and $C = 3$, because for any k each point of E^2 is covered by at least one, and by not more than three squares $P^r_{\varepsilon_k m}$ for some m and r. But on these squares $1/2 \leqslant u^r_{km} \leqslant 1$. The level sets $u^r_{km} = 1/2$ were constructed especially for this purpose.

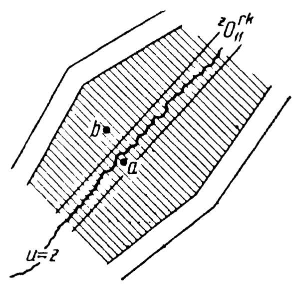


Figure 7. The bands are constructed at the nth stage. In the shaded area u differs from the value on the level $u = z$ (whose approximation is the middle band ${}^zO^{rk}_{11}$) by less than $\varepsilon/2$.

The condition 5) will be fulfilled if $g^r_{k'm'} \subset g^r_{km}$ because

$$g^r_{k'm'} = \bigcup_{i=k'}^{\infty} Q^{ri}_{\varepsilon_{k'}m'} \subseteq \bigcap_{i=k'+1}^{\infty} \hat{Q}^{ri}_{\varepsilon_{k'}m'} \subseteq \bigcap_{i=k'+1}^{\infty} {}^{x_{m'}}O^{ri}_{km},$$

that is, the set $g^r_{k'm'}$ is contained entirely in the level set of u^r_{km}. If $g^r_{k'm'} \subset R^2 \setminus g^r_{km}$, then $u_{k'm'} = 0$ on $g^r_{k'm'}$. The boundaries of g^r_{km} and $g^r_{k'm'}$ do not intersect, by their definition. Each of these sets is a region, and hence there can occur no other cases.

The condition 6) is also satisfied. This is obvious for the sets of the levels 0 and 1. (It is easy to see that each of the remaining level sets of the function u^r_{km} is obtained as the intersection of a sequence of closed polygonal bands, and it is, therefore, connected, and divides the plane into two parts, one containing the set where $u^r_{km} = 0$, the other one where $u^r_{km} = 1$.) The boundaries of the level sets of u^r_{km} that contain $g^r_{k'm'}$, divide the plane into not more than three parts because they are obtained as the intersection of sequences of closed polygonal Θ-type bands. The boundaries of the remaining level sets of u^r_{km} (with the exception of the 0 level set for which 6) is trivial) coincide with exactly these sets because none of such level sets contains points of the open set

$$\bigcup_{\substack{i,j \\ g^r_{k_i m_j} \subset g^r_{km}}} g^r_{k_i m_j},$$

which is everywhere dense in g^r_{km}, and consists of all points of g^r_{km} that belong to the sets of higher rank of the same system.

This completes the proof of the fundamental lemma.

§2. Proof of Theorem 2

Let u^r_{km} be functions that satisfy the conditions of the fundamental lemma, g^r_{km} be sets on which the functions are positive, and let d_k and $0 \leqslant c \leqslant C$ be the constants occurring in that lemma. For the purpose of constructing the representation of a function of three variables in the form indicated in Theorem 2, we first decompose a function of two variables into an absolutely and uniformly convergent series of the functions u^r_{km}.

Lemma 1. *Suppose that we are given on the square E^2 a family F of continuous functions which form a compact in the uniform metric (i.e. the family consists of uniformly bounded and equi-continuous functions u, and is closed with respect to uniform convergence). Then every function $f \in F$ can be represented in the form*

$$f(x) = \sum_{k=1}^{\infty} \sum_{r=1}^{3} \sum_{m=1}^{m_k} a^r_{km}(f)\, u^r_{km}(x), \tag{1}$$

where the coefficients a^r_{km} are independent of x, depend continuously (in the sense of the uniform metric) on the $f \in F$ and are such that

$$|a^r_{km}(f)| \leqslant a_k, \quad \sum_{k=1}^{\infty} a_k < \infty,$$

where the a_k *depend only on the family* F.

For the proof of this theorem we need the following proposition.

Lemma on the approximation by means of a linear combination of functions of rank k. *Let* $f(x)$ *be a continuous real function on* E^2, *and let*

$$\max_{x \in E^2} |f(x)| \leqslant M.$$

Let k *be a positive integer, and*

$$\max_{\rho(x, y) \leqslant d_k} |f(x) - f(y)| \leqslant \delta_k.$$

Then one can determine coefficients b_m^r, *independent of* x, *such that*

$$f(x) = S(x) + R(x), \tag{2}$$

where

$$S(x) = \sum_{r=1}^{3} \sum_{m=1}^{m_k} b_m^r u_{km}^r(x), \tag{3}$$

$$|R(x)| \leqslant \left(1 - \frac{c}{C}\right) M + \delta_k. \tag{4}$$

Hereby one can select the b_m^n *so that they depend continuously (in the sense of the uniform metric) on* $f(x)$, *and satisfy the inequality* $|b_m^r| \leqslant M/C$.

Proof. We pick a point x_{km}^r in each one of the sets g_{km}^r, and set $b_m^r = f(x_{km}^r)/C$. Obviously, the b_m^r depend continuously on f and $|b_m^r| \leqslant M/C$. Next we will show that the inequality (4) is fulfilled at each point $x \in E^2$. The $R(x)$ is determined by means of (2) and (3) for the given choice of b_m^r. Let us keep the arbitrary point $x \in E^2$ fixed. From the properties 2) and 3) of the functions u_{km}^r (see the fundamental lemma) it follows that at most three of the functions u_{km}^r, for a given k, will be different from zero at each point x, and these will correspond to different r. Suppose that for the given point x these functions are $u_{km_r}^r$ $(r = 1, 2, 3)$. Then, for the given point x, we have

$$S(x) = \sum_{r=1}^{3} b_{m_r}^r u_{km_r}^r(x) = \frac{1}{C} \sum_{r=1}^{3} f(x_{km_r}^r) u_{km_r}^r(x).$$

Let us suppose at first that $x_{km_r}^r$ $(r = 1, 2, 3)$ and x were selected so fortunately that they coincided: $x_{km_r}^r = x$ $(r = 1, 2, 3)$. Then $s(x)$ would be

$$S'(x) = \frac{1}{C} \sum_{r=1}^{3} f(x) u_{km_r}^r(x) = \frac{f(x)}{C} \sum_{r=1}^{3} u_{km_r}^r(x) \tag{5}$$

and $R(x)$ would be, correspondingly,

$$R'(x) = f(x) - S'(x). \tag{6}$$

But from the requirement 4) of the fundamental lemma it follows that

$$0 < c \leqslant \sum_{r=1}^{3} u^r_{km_r}(x) \leqslant C.$$

Therefore, we have the following estimate for $R(x)$,

$$|R'(x)| = |f(x) - S'(x)| = |f(x)|\left(1 - \frac{c}{C}\right) \leqslant M\left(1 - \frac{c}{C}\right). \tag{7}$$

The same estimate for $R'(x)$, defined by the equations (5) and (6) holds, obviously, also without the hypothesis that $x^r_{km_r} = x$ $(r = 1, 2, 3)$. In order to appraise $R(x)$ in the general case, we consider

$$|R(x) - R'(x)| = |S(x) - S'(x)| =$$

$$= \frac{1}{C} \left| \sum_{r=1}^{3} [f(x^r_{km_r}) - f(x)]\, u^r_{km_r}(x) \right| \leqslant \frac{1}{C} \sum_{r=1}^{3} |f(x^r_{km_r}) - f(x)|\, u^r_{km_r}(x).$$

Since (see condition 2) of the fundamental lemma) the diameter of the region g^r_{km} is less than d_k, we have that

$$|R(x) - R'(x)| < \frac{1}{C}\, \delta_k \sum_{r=1} u^r_{km_r}(x)$$

or, on the basis of property 4) of the fundamental lemma, that

$$|R(x) - R'(x)| < \delta_k.$$

This, in combination with (7), establishes the lemma.

Proof of Lemma 1. Let $f \in F$ be a real function continuous on E^2, and let

$$\sup_{x \in E^2,\, f \in F} |f(x)| \leqslant M = M_0, \qquad \sup_{\substack{x \in E^2,\, y \in E^2,\, f \in F \\ \rho(x,\, y) < d_k}} |f(x) - f(y)| = \delta_k.$$

As $k \to \infty$, $\delta_k \to 0$. Therefore, one can select a $k_1 = k_1(F)$ so large that $\delta_{k_1} < cM_0/2C$. Applying the lemma on the approximation, with $k = k_1$, and assuming that $a^r_{k_1 m} = b^r_m$, we obtain

$$f(x) = \sum_{r=1}^{3} \sum_{m=1}^{m_{k_1}} a^r_{k_1 m}(f)\, u^r_{k_1 m}(x) + R_1(x);$$

moreover

$$\sup_{x\in E^2,\, f\in F} |R_1(x)| \leqslant M_0\left(1-\frac{c}{C}\right)+\delta_{k_1} < M_0\left(1-\frac{c}{2C}\right),$$

where the $a^r_{k_1m}$ depend continuously on $f \in F$, and

$$|a^r_{k_1m}| < \frac{M_0}{C} = \frac{M}{C}.$$

Setting $1 - c/2C = \theta$, and $\theta M_0 = M_1$, we obtain

$$\sup_{x\in E^2,\, f\in F} |R_1(x)| < M_1.$$

It is obvious that the $R_1(x)$ that correspond to all possible $f \in F$, form a compact F_1, as a continuous image of a compact. In particular, these $R_1(x)$ are uniformly bounded and equi-continuous. Furthermore, each function $R \in F_1$ depends continuously on the corresponding function $f \in F_0$. Let us introduce the notation

$$\sup_{\substack{x\in E^2,\, y\in E^2,\, R_1\in F_1\\ \rho(x,\, y)\leqslant d_k}} |R_1(x) - R_1(y)| = \delta'_k.$$

We can repeat the preceding argument, and in conclusion obtain a $k_2 = k_2(F)$ such that

$$R_1(x) = \sum_{r=1}^{3}\sum_{m=1}^{m_{k_2}} a^r_{k_2m}(R_1)\, u^r_{k_2m}(x) + R_2(x),$$

where

$$\sup_{x\in E^2,\, f\in F} |R_2(x)| < \theta M_1 = \theta^2 M$$

and the $a^r_{k_2m}$ depend continuously on $R_1 \in F_1$, and, hence, on $f \in F_0$. Furthermore,

$$|a^r_{k_2m}| < \frac{M_1}{C} = \frac{M}{C}\,\theta.$$

Continuing in the same way, we obtain the sequence

$$R_n(x) = \sum_{r=1}^{3}\sum_{m=1}^{m_{k_{n+1}}} a^r_{k_{n+1}m}(R_n)\, u^r_{k_{n+1}m}(x) + R_{n+1}(x);$$

moreover

$$\sup_{x\in E^2,\, f\in F} |R_{n+1}(x)| < \theta M_n = \theta^{n+1}M, \tag{8}$$

where the $a^r_{k_n m}$ depend continuously on $f \in F$, and

$$|a^r_{k_{n+1}m}| < \frac{M_n}{C} = \frac{M}{C}\,\theta^n \tag{9}$$

($n = 0, 1, 2, \ldots,$ if we use the notation $R_0(x) = f(x)$).

Let us introduce the notation

$$S_n(x) = \sum_{i=0}^{n-1}(R_i(x) - R_{i+1}(x)) = \sum_{i=0}^{n-1}\sum_{r=1}^{3}\sum_{m=1}^{m_{k_{i+1}}} a^r_{k_{i+1}m}u^r_{k_{i+1}m}(x). \tag{10}$$

Then it is obvious that

$$f(x) = S_n(x) + R_n(x).$$

From the inequalities (8) and (9) it can be seen that the sequence $S_n(x)$ ($n = 1, 2, \ldots$) converges to $f(x)$ absolutely and uniformly, and that $|a^r_{k_i m}| < a_{k_i} = M\theta^{i-1}/C$ ($i = 1, 2, \ldots$).

This proves the lemma, since one may set $a^r_{km} = 0$ when $k \neq k_i$ ($i = 1, 2, \ldots$) and then obtain (1) from (10).

In the proof of Theorem 2 use is made of the following result.

Lemma 2. *The space of the components of the level sets of the function*

$$F_r(x, y) = \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{m=1}^{m_k} u^r_{km}(x, y)$$

is a tree with a branch point index not greater than three.

Every function

$$f_r(x, y) = \sum_{k=1}^{\infty}\sum_{m=1}^{m_k} a^r_{km}u^r_{km} \tag{*}$$

is constant on each component of a level set of the function F_r if the a^r_{km} are such constants that the series (*) converges uniformly and absolutely.

Proof of Lemma 2. Let r be fixed. First, let us prove that all the components of the level sets of the function $F_r(x,y)$ are 1) components of the level sets of the function u^r_{km} ($k = 1, 2, \ldots;\ m = 1, \ldots, m_k$), 2) boundaries of such components, 3) separate points which are intersections of sequences of the sets $g^r_{km'_k}$ ($k \to \infty$).

Let us pick a point a in the plane. The point a belongs either to an infinite number of the sets g^r_{km}, or there exists a "last rank" $k_0 \geqslant 0$ after which the point a does not belong to any g^r_{km} $(k > k_0)$.

Let us consider the first case. We will prove that such a point is a component of the level sets of the function F_r. Suppose that the point a belongs to an infinite sequence $\{g^r_{k_i m_i}\}$. From the condition 3) of the fundamental lemma it follows that the k_i are all distinct. We shall assume that $k_{i+1} > k_i$. One can easily deduce from the fundamental lemma (requirements 2) and 5)), that if the sets g^r_{km} and $g^r_{k'm'}$ intersect, and if $k' > k$, then $g^r_{k'm'} \subset g^r_{km}$. In the proof of the fundamental lemma given above, this result is obtained automatically (see the proof of the fact that the requirement 5) is fulfilled). Therefore, we have a sequence of inscribed sets $g^r_{k_i m_i} \supset g^r_{k_{i+1} m_{i+1}} \supset \ldots \ni a$. In this connection, $\bigcap_{i=1}^{\infty} g^r_{k_i m_i} = a$, since the diameters of the sets $g^r_{k_i m_i}$ tend to zero as $i \to \infty$ (requirement 2) of the fundamental lemma).

On the boundary M_i of each set $g^r_{k_i m_i}$, the value of the function F_r is less than that at the point a. Indeed, all the functions u^r_{km} $(k \geqslant k_i)$ are zero on M_i (this is a direct consequence of the requirements 2) and 5) of the fundamental lemma), while all the functions u^r_{km} $(k < k_i)$ take on the same values as at the point a (requirement 5)). At the point a, however, all the functions $u^r_{k_j m_j}$ $(j \geqslant i)$ are positive, and, therefore, $F_r(a) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u^r_{km}(a)$ is greater than F_r on M_i.

But each continuum that contains a, intersects some set of the M_i because $\bigcup_{i=1}^{\infty} M_i$ separates a from all the points of $R^2 \setminus a$ (Figure 8). This means that on each continuum that contains a one can find a point b where $F_r(b) \neq F_r(a)$, but this indicates precisely that a is a component of the level sets of the function F_r.

Now, let us consider the second case. Suppose the point $a \in g^r_{k_0 m_0}$ does not belong to any g^r_{km} $(k > k_0)$. Then a will belong to a continuum K, the set of a nonzero level z of the function $u^r_{k_0 m_0}$.

Let us assume at first that K does not contain the regions g^r_{km} $(k > k_0)$. Then $0 < z < 1$. We will prove that K is a component of a level set of the function F_r.

Let us select two sequences z^+_i and z^-_i $(i = 1, 2, \ldots)$ which converge to z from above and from below, and which are such that the sets M^-_i and M^+_i of the levels z^-_i and z^+_i $(i = 1, 2, \ldots)$ of the function $u^r_{k_0 m_0}$ do not contain the regions g^r_{km} and $0 < z^-_i < z < z^+_i < 1$. This can be done because

$0 < z < 1$ and the regions g^r_{km} constitute a denumerable set. The continua M^+_i and M^-_i, obviously, separate K from the points where $u^r_{k_0 m_0}$ is greater than z^+_i and less than z^-_i, and all of them together, i.e. $\bigcup_{i=1}^{\infty} (M^+_i \cup M^-_i)$, separate K entirely from all points of $R^2 \setminus K$, since at every point of $R^2 \setminus K$, $u^r_{k_0 m_0}$ is greater than some z^+_i or less than some z^-_i. On K, as well as on the sets M^+_i and M^-_i, the function F_r does not change since all the terms with u^r_{km} $(k > k_0)$ are zero in view of the assumption on the absence on K, M^+_i and M^-_i of the sets g^r_{km}. But the values of F_r on K, on M^+_i and on M^-_i are different, because all terms u^r_{km} with $k < k_0$ are the same on these continua (requirement 5)), all the terms u^r_{km} with $k > k_0$ are equal to zero, while the function $u^r_{k_0 m_0}$ is equal to z on K, to z^+_i on M^+_i, and to z^-_i on M^-_i.

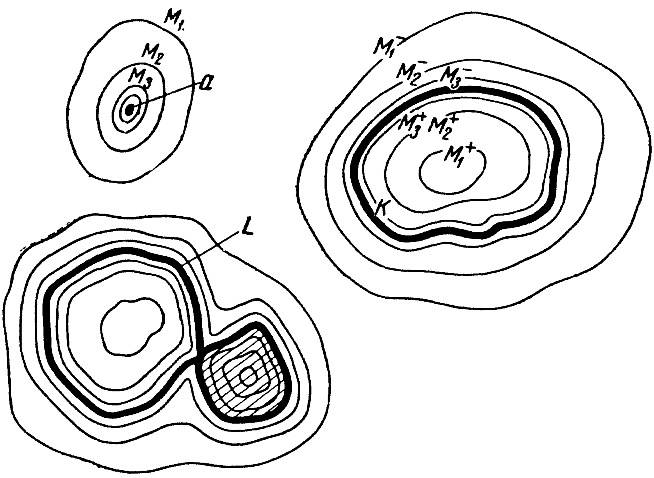


Figure 8. Representation of all types of components. In the third case $z \neq 1$. The $g^r_{k'm'}$ are lined. The case $z = 1$ is left to the reader.

Each continuum $M \neq K$, but containing K, intersects $\bigcup_{i=1}^{\infty} (M^+_i \cup M^-_i)$ (Figure 8). Therefore it has points where F_r differs from the values of F_r at the points of K. This means that K is a component of the set of levels of the function F_r.

In the remaining case the proof is analogous to the one given above, and we will only indicate it. If the set $K \ni a$ of the level $u^r_{k_0 m_0} = z$ contains $g^r_{k'm'}$, then the component of the level set of the function F_r that contains a will be L, the boundary of K (Fig. 8). Actually, the point a does not belong to $g^r_{k'm'}$ (since k_0 is the "last rank"). The boundary of K divides the plane into no more than three parts (requirement 6)). First, suppose that $z \neq 1$. Then in two of these parts $u^r_{k_0m_0}$ will take on values greater and less than z, while in the third part $g^r_{k'm'}$, $u^r_{k'm'}$ is positive. The point a cannot lie in any of these parts but lies on the boundary of K. On the continuum L, the function F_r is constant, because all the functions u^r_{km} are constant (requirements 5) and 6)). In order to prove that L is a component of the level set of the function F_r, it is necessary to separate it by means of continua, with values of F_r, from all points of $R^2\setminus L$. For this it is necessary to use sets of levels near zero of the function $u^r_{k'm'}$ and sets of levels close to z of the function $u^r_{k_0m_0}$ (Figure 8).

The remaining case, $z = 1$, is even simpler because the boundary of the set $u^r_{k_0m_0} = 1$ divides the plane into two parts only (this is a direct consequence of the construction of the functions u^r_{km}, but it can also be deduced from requirements 2) and 6) of the fundamental lemma).

The structure of the components of a level set for the function F_r has thus been explained. Not a single one of them divides the plane into more than three parts. It follows (Appendix, Theorem 3) that the tree of the function F_r consists of points whose branching index does not exceed 3.

In order to complete the proof of Lemma 2, we note that all the functions u^r_{km} are constant on each component of the level sets of F_r. This implies the truth of the second assertion of the lemma.

Theorem 2. *Every real function $f(x_1,x_2,x_3)$ that is continuous on E^3 can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1, x_2), x_3],$$

where h_i and φ_i are continuous functions, the functions h_i are defined on the product $\Xi \times E^1$ of the tree by the interval E^1, while the $\varphi_i(x_1,x_2)$ are defined on the square E^2, and have for their values points of Ξ. Here Ξ is a tree whose points have branching indices not greater than 3.

Proof. A function $f(x_1, x_2, x_3)$ of three variables can be considered as a family of functions of two variables that depends on the third variable as a parameter: $f_{x_3}(x_1,x_2)$, where the function $f_{x_3}(x_1,x_2)$ is defined for each x_3 on a single square $0 \leqslant x_1, x_2 \leqslant 1$, and at a point (a,b) is equal to $f(a,b,x_3)$. Obviously, each of the functions $f_{x_3}(x_1,x_2)$ is continuous and depends continuously (in the sense of the uniform metric) on the parameter x_3 $(0 \leqslant x_3 \leqslant 1)$. Therefore, the family of functions $f_{x_3}(x_1,x_2)$ forms a compact. Hence, we can apply the Lemma 1 and obtain

$$f_{x_3}(x_1, x_2)=\sum_{k=1}^{\infty}\sum_{r=1}^{3}\sum_{m=1}^{m_k} a^r_{km}(x_3)\,u^r_{km}(x_1, x_2).$$

Since $|a^r_{km}(x_3)| \leqslant a_k$, and $\sum\limits_{k=1}^{\infty} a_k < \infty$, it follows that each of the series

$$f^r_{x_3}(x_1, x_2)=\sum_{k=1}^{\infty}\sum_{m=1}^{m_k} a^r_{km}(x_3)\,u^r_{km}(x_1, x_2) \quad (r=1,\ 2,\ 3)$$

converges absolutely and uniformly. (But by the fundamental lemma only one of the u^r_{km} $(m = 1,\dots,m_k)$ is different from zero at any given point.) We shall show that $f^r_{x_3}(x_1, x_2)$ depends on x_3 continuously (in the same sense).

Indeed, suppose $\varepsilon > 0$. We can select N so large that $\sum\limits_{k=N}^{\infty} a_k < \varepsilon/4$. Since the $a^r_{km}(x_3)u^r_{km}(x_1,x_2)$ depend continuously on x_3, the same thing must be true for the finite sum. Hence there exists a $\delta > 0$ such that if $|y - z| < \delta$ then

$$\sup_{x_1, x_2\in E^2}\left|\sum_{k=1}^{N-1}\sum_{m=1}^{m_k} a^r_{km}(y)\,u^r_{km}(x_1, x_2)-\sum_{k=1}^{N-1}\sum_{m=1}^{m_k} a^r_{km}(z)\,u^r_{km}(x_1, x_2)\right|<\frac{\varepsilon}{4}$$

$$(r=1,2,3).$$

But since

$$\sup_{x_1,x_2\in E^2}\left|\sum_{k=N}^{\infty}\sum_{m=1}^{m_k} a^r_{km}(y)\,u^r_{km}(x_1, x_2)-\sum_{k=N}^{\infty}\sum_{m=1}^{m_k} a^r_{km}(z)\,u^r_{km}(x_1, x_2)\right|\leqslant 2\sum_{k=N}^{\infty} a_k<\frac{\varepsilon}{2},$$

we find that for $|y - z| < \delta$, it is true that

$$\sup_{x_1, x_2\in E^2}|f^r_y(x_1, x_2)-f^r_z(x_1, x_2)|<\varepsilon \quad (r=1,\ 2,\ 3).$$

Now we apply Lemma 2 and see that for any given x_3, each of the functions $x_3 \in [0,1]$ is constant on each component of the level set of one of the constructed functions $F_r(x_1,x_2) = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}\sum\limits_{m=1}^{m_k} u^r_{km}(x_1,x_2)$ which does not depend on $f(x_1,x_2,x_3)$.

Let us consider (see Appendix) the tree of components of the level sets of the function $F_r(x_1,x_2)$. The mapping $t(a) = \varphi_r(x_1,x_2)$ associates with each point x of the square $E^2 = A$ a point φ_r of the tree T^r which represents the component t of the level set of $F_r(x_1, x_2)$ that contains (x_1, x_2). We can consider this mapping as a function $\varphi_r(x_1, x_2)$ defined on the square and with values from the tree. If one wishes, one can realize the tree on a plane. This mapping can then be written with the aid of two real functions defined on the square. The mapping $t(a)$ is continuous. The functions $f^r_{x_3}(x_1, x_2)$ generate on T^r functions $f^r_{x_0}(\varphi_r)$ which are equal to the values of $f^r_{x_3}(x_1, x_2)$ at any point of the component t of the counterimage of φ_r on E^2. Because of Lemma 2, this value is the same at all points of this component. It is obvious that the obtained functions $f^r_{x_3}(\varphi_r)$ are continuous on T^r and depend continuously on x_3. Therefore, one may consider the family $f^r_{x_3}(\varphi_r)$ $(x_3 \in [0, 1])$ as a continuous real function $f^r(x_3, \varphi_r)$ on the product of the tree by the interval of variation of x_3:

$$f^r_{x_3}(x_1, x_2) = f^r(x_3, \varphi_r(x_1, x_2)).$$

From the three trees T^r $(r = 1, 2, 3)$ we can compose a single tree Ξ. By Lemma 2, each of the three trees consists of points whose branching indices are 1, 2 or 3. The tree Ξ, obviously, can be constructed so that it has the same property. Each of the functions $f^r(x_3, \varphi_r)$ $(r = 1, 2, 3)$ can be extended continuously over the product of the entire tree Ξ by the interval (it does not matter in what way this is done). Let us denote this extension by $h_r(\varphi_r, x_3)$ $(r = 1, 2, 3)$. From the relation (1), Lemma 1, we obtain in this notation

$$f(x_1, x_2, x_3) = \sum_{r=1}^{3} h_r[\varphi_r(x_1, x_2), x_3].$$

This completes the proof of Theorem 1.

PART II

Proof of Theorem 3

We shall now construct the tree $X \subset E^3$ mentioned in Theorem 3. This tree is to be homeomorphic to the universal tree Ξ which does not have points whose branching index is greater than 3. The latter tree, as is well known (see Appendix, Theorem 5), can be obtained by attaching branches. More precisely, Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}, \quad \Delta_n \subset \Delta_{n+1},$$

where Δ_n is a finite tree (curved complexes), Δ_1 is a simple arc and Δ_{n+1} is obtained from Δ_n by attaching at the point ρ_n (which is not a branch point) simple arcs σ_n (Figure 9). We note that the set of points ρ_n that are now branch points of Ξ, is at most denumerable. Everything that pertains to the abstract tree Ξ will be denoted by Greek letters, while the corresponding items of its realization X will de designated by Latin letters. The realization of X will be constructed in the form

$$X = \overline{\bigcup_{n=1}^{\infty} D_n}, \quad D_n \subset D_{n+1},$$

where the D_n are segment complexes in a three-dimensional space; the homeomorphism between Ξ and X will be constructed as a continuation of the homeomorphisms Δ_n and D_n.

It is known that in order for X to be a realization of Ξ (and, hence, to be a tree), it is sufficient that the following conditions be fulfilled (see Appendix, Lemmas 10 and 11):

α) Each newly constructed branch s_n, except for its base, must lie entirely inside the open, still empty, simplex T_n. Furthermore, for all twigs s_m attached to s_n $(p_m \in s_n)$ later $(m > n)$ $\overline{T_m} \subset T_n$ (Figure 9).

On Figure 9, and in Menger's work ([3], Chapter X), where the tree X lies in a plane, the simplexes T are triangles. In our case they are tetrahedra. This makes no essential difference.

β) The simplexes T_n must be sufficiently small: the diameters of the T_n tend to zero when $n \to \infty$.

γ) The points p_n at which the new branches are attached may not have been earlier branch points or endpoints for D_n.

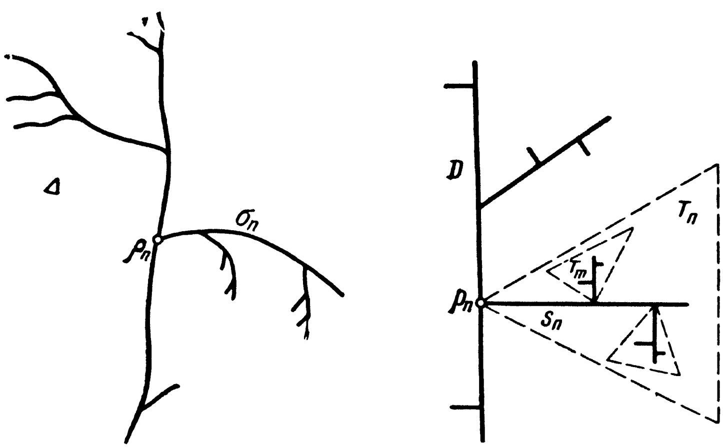


Figure 9. Finite trees: " abstract ", curved tree Δ, and its realization as a complex D.

In the sequel (§§ 3-7), in the construction of D_n and X, we will always be able to choose the points p_n and the direction of the segments s_n with sufficient freedom: each time the forbidden points or directions will have an everywhere dense complement. The length of the s_n will always be chosen sufficiently small. The conditions α), β), and γ) can, therefore, be assumed to have been satisfied at each step. In order not to complicate the future presentation, we will not mention this in the sequel. We assume that by attaching each branch s_n we construct the corresponding tetrahedron T_n, and will not worry about the fulfillment of the conditions α), β), and γ).

In order that the obtained tree X may satisfy the conditions of Theorem 3, i.e. in order that each continuous function of the given family may be represented as the sum of functions of coordinates, it is necessary to select the p_n and s_n with certain restrictions. For the precise formulation of these restrictions, we need several new concepts which are presented in the next section.

§3. Fundamental definitions. Inductive properties 1-4

In a three-dimensional space* let K be a finite set of segments or straight lines. These segments (straight lines) are not to be parallel to the coordinate planes.

Definition 1 (Figure 10). A *zigzag* (certain type of broken line) is a system of points $a_0 \neq a_1 \neq \ldots \neq a_{n-1} \neq a_n$ of K, such that the segments $a_{i-1}a_i$ $(i = 1, \ldots, n-1)$ are perpendicular to the coordinate axes x_{α_i} and $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \ldots \neq \alpha_n$. The segments $a_{i-1}a_i$ are called *links* of the zigzag. If $a_0 = a_n$, the zigzag is said to be closed.

One should visualize the zigzag in the following way. The beginning a_0 is a point of K. We choose the first direction α_1. The plane that passes through a_0 and is perpendicular to the axis x_{α_1} (we shall refer to it as the " plane of the coordinate direction α_1") intersects K at a point a_1. We shall say that it leads from a_0 to a_1. In exactly the same way the link a_1a_2 lies in the plane of the direction α_2 $(\neq \alpha_1)$ so that at a_1 there occurs a break. At the point a_2, the direction again changes to α_3 $(\neq \alpha_2)$ and we arrive at the point a_3, and so on until we get to a_n, the end of the zigzag.

By somewhat modifying the described process we obtain the generating

* In § 3-7 the number of dimensions could be $\geqslant 2$. The graphs correspond to the two-dimensional case.

scheme that was defined in note [1]. A more descriptive definition will be given here.

Definition 2 (Figure 11). The beginning of the generating scheme is the point $a_0 \in K$.

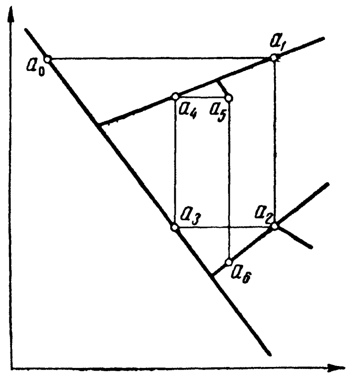


Figure 10. The zigzag $a_0a_1a_2a_3a_4a_5a_6$.

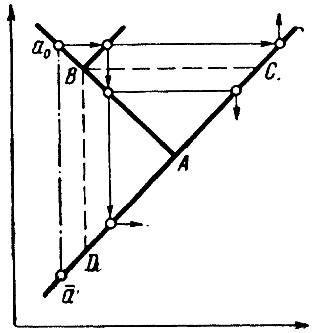


Figure 11. A generating scheme from the point a_0. If one includes the point $\bar{a}$, then the obtained double generating scheme will be of the class of the point a_0.

The beginning is also called the end of rank 0. We choose a coordinate direction α_0 and draw through the point a_0 the plane of this direction. In general, it will have several points of intersection with K in addition to the point a_0. We shall call this plane a plane of rank 1, and these points, ends of rank 1. The plane of rank 1 *leads* from a_0 into each of the ends of rank 1.

Next, this process is continued. At each end a of rank n we select a coordinate direction α different from the one along which we arrived at this end.* Through a we pass the plane of this direction. If this plane does not pass through any other point of K besides a, we do no more to this point a; it is called a free end. If, however, a is not a free end, then we obtain points of intersection of the plane with K, which are called ends of rank $n + 1$. In this manner the constructed plane of rank $n + 1$ leads away from the nonfree end of rank n and leads to ends of rank $n + 1$.

If this process terminates, i.e. if all the ends of some rank N are free ends, and if all ends of all ranks as pairwise distinct,** then the entire

* That is, different from the direction of the plane of rank n at whose intersection with K the point a lies.

** This means that in the construction we do not arrive at the same point twice.

structure is called a generating scheme which leads from the point a_0 in the direction α_0. N is called the rank of the scheme.

In this manner, a *generating scheme* (or system) consists of a beginning a_0, of "supporting" planes of different ranks, and of ends of different ranks.

We will need a certain generalization of generating systems, a double generating system. It differs from the simple one defined above only in that from some of its points (ends) one draws two planes, and not just one. In this way, all three directional coordinate planes that pass through such a point, can be supporting planes in a double scheme if one of these planes leads into, and the other two away from the point. Double systems can be obtained, for example, by combining simple ones which have only the beginning in common, or by connecting to some nonfree end a of a simple scheme A a generating system B, for which a is the beginning, and which has no common points with A except a.

Every free end a of a double (or simple) generating system can be connected to the beginning a_0 by a unique zigzag all of whose points are ends of a scheme, and all of whose links lie in the supporting planes of the scheme. If there were several such zigzags, then the ends of the scheme could not be pairwise distinct. The indicated zigzags are called zigzags of the scheme. They are finite, not closed, and do not contain closed parts.

Definitions of stability. We shall say that two zigzags (generating schemes) on K are of the same type if their points can be put into a one-to-one reciprocal correspondence in such a way that corresponding points lie on the same segments (straight lines) of K,* while the corresponding links are perpendicular to the same coordinate axis.

We shall say that the zigzag $a_0 \ldots a_n$ is not longer than the zigzag $b_0 \ldots b_m$ $(m \geqslant n)$ if it is of the same type as a part $b_0 \ldots b_n$ of the second one.

A generating scheme A is not longer than a generating scheme B if one can set up a correspondence between their zigzags under which all zigzags of A are not longer than the corresponding zigzags of B. The types of the generating schemes which are not longer than a given one form a finite set.

We shall say that a generating scheme A that begins at a_0 is stable if a_0 has such a neighborhood that the generating schemes of the points of K that lie in this neighborhood are not longer than A. For example, the complex K of Figure 11 admits a generating scheme, beginning at any point

* No branch points can lie within a segment of a segment-like complex K. The complex of Figure 11 consists of 5 segments. This remark does not apply to the set of straight lines of K.

with an arbitrary first direction. Here, for any point, except for the branch points A, B, and the end points C, D of the zigzag that issue from B, the scheme is stable.

The zigzags of the same type produce a mapping of the set of all their beginnings (initial points) into the set of their ends. This mapping is linear and nondegenerate (because the segments of K are not perpendicular to the coordinate axes). We will make frequent use of these facts in what follows.

The set of all points of K which are vertices of zigzags that issue from the point a_0 are called the class of points accessible from a_0, or simply the class of the point a_0 on K. The class of a set of points is defined in an analogous way. We call attention to the fact that the class of a point, and hence the class of a denumerable set is a denumerable set. All generating schemes of a point a_0, and of points belonging to the class of a_0 lie in the same class.

Now we can formulate the inductive lemma which will be proved in §§4-9.

Let us return to the function f on the tree Ξ.

Suppose that ω_n is the upper boundary of the variation of the functions $f \in F$ on the component difference $\Xi \setminus \Delta_n$. As $n \to \infty$, $\omega_n \to 0$.

Indeed, if Ξ' is a realization of Ξ constructed (see Appendix, Theorem 5) on the plane, then F will give rise to a family F' of equi-continuous functions defined on the planar continuum Ξ'. Since the diameter (see condition β, and Figure 9) of the triangles T_n tends to zero when $n \to \infty$, and since every component $\Xi' \setminus \Delta'_n$ lies in the triangle T_m $(m > n)$, it follows that for large enough n the diameter of the component $\Xi' \setminus \Delta'_n$ will be so small that the oscillation of any function $f' \in F'$ will be arbitrarily small on every component. Therefore one can pick a sequence

$$n_1 < n_2 < \ldots < n_r < \ldots,$$

so that $\omega_n \leqslant 1/r^2$ when $n > n_r$.

We shall next list the *inductive properties* of the tree D_n, of the homeomorphism of Δ_n on D, and of the functions $f_k^m(x_k)$ $(m \leqslant n;\ k = 1, 2, 3)$. Here the tree D is a realization of Δ_n. D lies in a three-dimensional cube of a segment-like complex whose segments are not perpendicular to the coordinate axes.

1. Let A be the set of points of D_n which are images of the branch points* of Ξ that lie on Δ_n. Let K_n be the set of straight lines whose segments form D_n, and let C_n be the class of the set of vertices of the

* More precisely, one should say of the "points ρ_m" because Theorem 4 is not proved in the Appendix. In the sequel, branch points can be taken as the points ρ_m and p_m.

closed zigzags on K_n.

Then:

a) C_n is at most denumerable,

b) C_n does not intersect A_n (and hence not the class A_n on K_n),

c) no two points of A_n belong to the same class on K_n.

2. On D_n there is a finite number of simple generating schemes such that from any point $a_0 \in D_n$ one can start in any direction to generate the scheme of one of the " canonic " type.

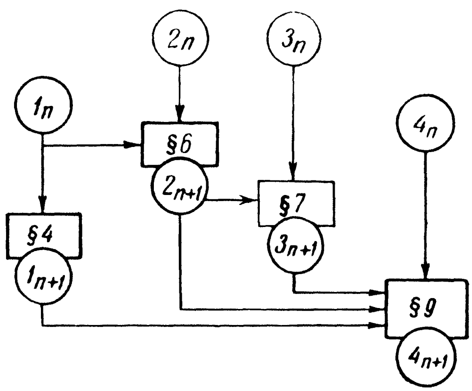
3. Every function $f \in F$ is representable on D_n in the form $f(x) = \sum_{k=1}^{3} f_k^n(x_k)$, where the x_k are the coordinates of the point $x \in D_n$, and the $f_k^n(x_k)$ are continuous functions which depend continuously on $f(x)$.

4. If $n_r < n \leqslant n_{r+1}$, then

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| < \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right)\frac{1}{r^2}.$$

Inductive lemma. *If the tree D_n, the homeomorphism of Δ_n on D_n, and the functions $f_k^m(x_k)$ $(k = 1,2,3;\ m \leqslant n)$ have the properties 1 to 4, then one can construct a tree D_{n+1}, a mapping of Δ_{n+1} on D_{n+1}, and functions $f_k^{n+1}(x_n)$, with the same properties, by attaching to the point p_n a branch-segment s_n that is not perpendicular to the coordinate axes.*

Scheme of proof:



i_n indicates the property i of the tree D_n, of the homeomorphism Δ_n onto D_n, or of the function $f_k^n(x_k)$. In the section that appears in any rectangle, the property i_{n+1} is derived from the properties that are connected with this section by means of arrows.

§4. Inductive preservation of property 1

We will assume that D_n has the property 1, and we will show what conditions have to be imposed on s_n in order that this property may be preserved on D_{n+1}. The conditions that one finds are not very restrictive: the direction may be chosen from an everywhere dense set of the second category;* the length can be arbitrarily small.

Let us now assume that on K_n, that is on D_n, to which there have been added rays which extend the segments D_n, the following conditions hold:

a) the class C_n of the points of closed zigzags is at most denumerable;

b) the points of closed zigzags of K_n are not accessible on K_n from the points of A_n which are the images on D_n of the branch points of Ξ;

c) no two points of A_n are such that one is accessible from the other on K_n.

Let us first restrict the selection of the direction of s_n in such a way that the condition a) is guaranteed on D_{n+1}. The number of the types of zigzags is at most denumerable for every choice of s_n, because the type is determined by the initial and successive straight lines of K_n and by the direction of the path, i.e. by a finite sequence of elements of a finite set. For each type there either is no closed zigzag, or there is one, or else all zigzags of the given type are closed. This follows from the linearity of the corresponding type of mapping of the initial straight line onto a finite one. In case that all zigzags of a type are closed, we say that a closed zigzag is stable. Obviously, it is sufficient that there be no closed zigzags on K_{n+1} in order that condition a) be satisfied on D_{n+1}.

Suppose that D_{n+1} has been constructed, and that the segment s_n is not perpendicular to the axes. The stable closed zigzags can occur only among zigzags which have a common point with the straight line l that supports s_n.

Let M be such a point. It can be taken for the beginning of a closed zigzag. Suppose that the equations of the straight line l in the system of coordinates in which the origin 0 has been translated to p_n are given as

$$x_2 = bx_1, \quad x_3 = cx_1,$$

where neither b nor c are zero, because the segment s_n is not parallel to the coordinate planes. For the sake of definiteness, let us assume that a closed zigzag issues from the point $M(x_0, bx_0, cx_0)$ in the direction x_1 and falls on l for the first time again exactly at the point M where it arrives from the direction x_2. Let the straight line at which the zigzag arrives at

* That is, from the complement of the sum of a denumerable number of nowhere dense sets.

the ith step, have the equations $x_2 = b_i x_1 + \beta_i$, $x_3 = c_i x_i + \gamma_i$. Neither one of the coefficients b_i, and c_i can be zero. The second point of the zigzag has the coordinates $x_0, b_1 x_0 + \beta_1$, $c_1 x_0 + \gamma_1$. If the second direction is, for example, x_2, then the coordinates of the third point will be

$$\frac{b_1 x_0 + \beta_1 - \beta_2}{b_2}, \quad b_1 x_0 + \beta_1, \quad c_2 \frac{b_1 x_0 + \beta_1 - \beta_2}{b_2} + \gamma_2.$$

In general the coordinates of each point depend linearly on x_0, and the coefficients are determined by the intermediate straight lines. We assume that the zigzag does not intersect l before it is closed. Then the last point will have the coordinates

$$l_1 x_0 + \lambda_1, \quad l_2 x_0 + \lambda_2, \quad \lambda_3 x_0 + \lambda_3$$

because the direction x_2 will lead to the point x_0, bx_0, cx_0, and one obtains $bx_0 = l_2 x_0 + \lambda_2$. For stable closure it is necessary that the equation be satisfied for all x_0, i.e. $b = l_2$ and $\lambda_2 = 0$. Hence, such a closed zigzag will be stable only if l lies in the plane $x_2 = l_2 x_1$. The corresponding directions l will be called forbidden directions.

If the zigzag closes after it has been on l several times, a necessary condition for stability is $b^i c^j = l_0$, where l_0 is some constant depending on K_n and on the type of the zigzag. Here i is the difference between the number of arrivals of the zigzag on l from the direction x_2 and the number of departures from l in this direction; j has a similar meaning for the direction x_3. If the direction of l is not a forbidden one (i.e. $b^i c^j \neq l_0$), then there can exist no closed zigzags of the considered type. Suppose that $(l_0 - 1)^2 + i^2 + j^2 \neq 0$. Then the directions l for which $b^i c^j = l_0$ form nowhere a nondense set (a curve) in the space of directions. Therefore all directions which are forbidden by some types of zigzag for which $(l_0 - 1)^2 + i^2 + j^2 \neq 0$, lie on a denumerable set of smooth curves and constitute an everywhere dense set of the second category of forbidden directions.

If, however, $i = 0$, $j = 0$, and $l_0 = 1$, then the closed zigzag will be stable for b and c arbitrary, and, in particular, if we direct l along the straight line q on whose segment $q_n \in D_n$ the point p_n lies, where the branch s_n is attached. It is true here that some, but not all, points of the zigzag (namely those lying on l and q) will run together. But it is easy to see that on K_n there is defined a stable zigzag and that the points of q will belong to its class. But on q there are points A_n. This yields a contradiction with the condition c) which is satisfied by D.

The final result is as follows: one can choose the direction from an everywhere dense set of the second category so that D_{n+1} will satisfy the condition a).

Let us now go over to the condition c). We consider two branching points of Ξ whose images lie on D_n (in A_n). The s_n must be chosen so that it will be impossible to connect the points from A_n in D_{n+1} by means of a zigzag. For zigzags which do not contain points of s_n, this is already so, because the condition c) is satisfied on D_n. The set of pairs of points A_n is denumerable. So is the set of all types of zigzags. Let us consider one of these types and one of the pairs. The requirement that a zigzag of this type connects these points leads to forbidden directions l for which such a connection can occur, and, just as in the preceding proof, all forbidden directions lie on a denumerable set of smooth curves. The condition b) leads to the same type of requirement for the points A_n and closed zigzags K_{n+1}.

We must now concern ourselves with the fulfillment of condition b) for the points of $A_{n+1} \setminus A_n$ (lying on s_n) and with condition c) for pairs of such points A_{n+1} of which at least one lies on s_n. Having selected in the manner described the direction l from the everywhere dense set of the second category (from the complement of the forbidden directions), we map σ_n on s_n.

Thus we have constructed K_{n+1}. Let us put on l the points of the class A_n. This denumerable set must not intersect the images of the branching points of Ξ on s_n. The same prohibition applies to the set of points of the closed zigzags on K_{n+1} and the classes (on K_{n+1}) of these points. The set of forbidden points or, as we shall say, the "taboo set" is at most denumerable because of the way in which l is chosen.

The requirements a) and b) will be fulfilled on D_{n+1}, while the requirement c) will hold on D_n if we do not map the branching points of Ξ into the forbidden points s_n. Such a mapping of s_n on l for which the requirement c) on D_{n+1} is also satisfied, will now be described. Here the segment s_n may be arbitrarily small. This fact will be used later.

Let us now assume that we have chosen s_n and its size. On s_n there is a taboo set (at most denumerable) which cannot contain the images a of branching points α of Ξ that lie on σ_n. The mapping must be homeomorphic, and we must take care that the points a are inaccessible to each other on K_{n+1}.

Let us arrange the branching points of Ξ on σ_n into a sequence $\alpha_1, \alpha_2, \ldots$. (The point ρ_n is not included in this sequence.) The denumerable set is everywhere dense on σ_n.* Therefore, a similar** mapping of this set on

* If this is not the case then we add to the points α some points σ_n.

** That is, an order preserving.

an everywhere dense set s_n can be extended to the homeomorphism σ_n on s_n. We still have to map the points α on s_n. Since there are no ends σ_n among the α_i, the images a_i of the α_i are distributed in the interval s_n' whose closure is s_n.

Let us consider on s_n a denumerable system of intervals δ_i^k, $1 \leqslant k < \infty$, $1 \leqslant i \leqslant i_k$, such that

1) for every k $\bigcup_{i=1}^{i_k} \delta_i^k = s_n'$,

2) each of the intervals $\delta_1^k, \ldots, \delta_{i_k}^k$ is less than ε_k; $\varepsilon_k \to 0$ when $k \to \infty$.

If each of the intervals contains at least one point α_j, then the points a_j $(j = 1, 2, \ldots)$ form an everywhere dense set s_n. Let us arrange all these intervals into one sequence δ_l $(l = 1, 2, \ldots)$.

Let us assume that the directions on σ_n and s_n have been selected so that ρ_n and p_n are the left endpoints.

First step. We pick a nonforbidden point a_1 on δ_1. It will be the image of a point α_1. The points of the class of a_1 form on s_n a denumerable set. We add this set to the taboo set.

Second step. The point α_1 divides σ_n into a left and a right part. Let α_{i_1} be a point α with smallest subscript in the left part, while α_{i_r} has the same meaning for the right part. The point a_1 divides the intervals δ into those that lie to the left of a_1, those that lie to the right of a_1, and those that contain a_1. Among the intervals that do not lie to the right of a_1, with a subscript greater than 1, we select the one with the smallest subscript. On it we pick a nonforbidden point to the left of a_1. This will be a_{i_1}, the image of α_{i_1}. We add to the taboo set all points of the class of a_{i_1}. We select from the remaining intervals δ which are not to the left of a_1, the one with the smallest subscript. In this interval we pick a nonforbidden point a_{i_r} to the right of a_1. We add to the taboo set the points of the class of a_{i_r}.

After the nth step, σ_n will be divided into 2^n intervals by the $2^n - 1$ points $\alpha_1, \alpha_{i_1}, \alpha_{i_r}, \alpha_{i_{11}}, \alpha_{i_{1r}}, \ldots, \alpha_{i_{\underbrace{rr\ldots r}_{n-1 \text{ times}}}}$.

The $(n+1)$st step. In each one of the 2^n intervals we pick a point with the smallest subscript and denote this subscript in the left-most interval by $i_{\underbrace{11\ldots 1}_{n \text{ times}}}$, then by $i_{\underbrace{11\ldots 1r}_{n-1 \text{ times}}}, \ldots,$ in the right-most one by $i_{\underbrace{rr\ldots r}_{n-1 \text{ times}}}$.

The mapping of these 2^n points $\alpha_{i_{11\ldots 1}}, \alpha_{i_{rr\ldots r}}$ on s_n takes place in exactly the same way as described in the second step. The image of α, the point a, is always picked in the interval δ which is not to the left of α_i, the left end of the interval (α_i, α_j) from which α was picked, and not to the right of α_j. Hereby one picks the interval with the smallest subscript from all the intervals δ having the given property. In the interval δ, the point a is picked between a_i and a_j from the nonforbidden points. Then one adds to the taboo set all the points of the class of the point a. After this one constructs the image of the next point α until the $(n + 1)$st step is completed.

The proofs that the mapping of the points α on a is defined after the performance of all steps for all α, that this is a similar (order preserving) mapping, and that by the thus generated homeomorphism σ_n and s_n retain the properties a), b), and c), can be accomplished without difficulty.

§5. Lemma on generating schemes

Before we start the proof of the possibility of preserving the inductive properties 2, 3, and 4, let us investigate more closely generating schemes of segment-like complexes K. It is immaterial whether these schemes are simple or double.

If one omits the beginning in a generating scheme, then the remainder can be considered as the set of intersecting generating schemes starting with the ends of the first rank (of the shortened system).

Lemma 1 (Figure 12). *If the shortened systems A_i of a given system A are stable, and the initial point a_0 is not a branch point of K, then A is stable.*

Proof. Let $\varepsilon_1 > 0$ be such a number that an ε_1-shift* of the initial points a_i of the shortened schemes will not lengthen these systems (see definition of stability in §3). Furthermore, from the stability of A_i it follows that the a_i are not branching points of K. Since the complex K is a closed set, there exists an $\varepsilon_2 > 0$ such that the plane, which is parallel to the first plane of the scheme A and which is at a distance of at least ε_2 from it, intersects only those segments of K which contain a_0 and the points a_i.

By taking $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$, we obtain an ε-neighborhood of the point a_0. The existence of this neighborhood proves the stability of the scheme A.

* We recall that the distance between the points (x_1, x_2, x_3) and (x_1', x_2', x_3') is $\max_{1 \leqslant i \leqslant 3} (|x_i - x_i'|)$.

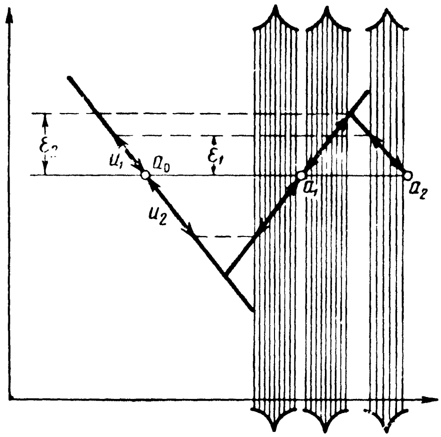


Figure 12. The generating scheme $a_0a_1a_2$ has rank 1. The layers of generating schemes of semineighborhoods are shaded.

Lemma 2. *If no vertex of a generating scheme is a branching point, then the generating scheme is stable.*

Proof. The proof is accomplished by induction. If the rank of the scheme is zero and the point a_0 is a free end, and not a branching point of K, then, obviously, there exists in K a neighborhood of the point a_0, which is composed of points with the same property (see Figure 12, where the points a_1 and a_2 are shown with the mentioned neighborhoods of stability). If the assertion of the Lemma 2 has been proved for a scheme of rank n, then its truth for a scheme of the next higher rank follows from Lemma 1.

Lemma 3. *Suppose that the generating scheme A with initial point a is stable. Then for every positive ε there exists a positive number δ such that every supporting plane that corresponds to the scheme A and leads away from the point b of B is at a distance not greater than ε from the corresponding plane of the scheme A, provided that the initial point b is nearer than δ to the initial point a.*

Proof. The generating scheme A has a finite number of supporting planes $\Pi_i^r(a)$ of each direction $r = 1, 2, 3$.

For the scheme B which leads away from the point b in the interval of stability of the scheme A, there are defined planes, points, and zigzags of the scheme B that correspond to planes, points, and zigzags of the scheme A. (The converse is in general not true, because the zigzags of the scheme B can terminate earlier.)

Let us consider the planes $\Pi_i^r(b)$. (This is the notation for the plane which corresponds to the plane $\Pi_i^r(a)$ in the scheme A.)

The coordinate x_r is the same for all the points $\Pi_i^r(b)$; it depends linearly on any coordinate of the initial point b. It follows from this and from the finiteness of the number of supporting planes of the scheme A, that for every point b in a sufficiently small neighborhood of the point a all planes of the scheme B are nearer than ε to the corresponding planes of the scheme A.

We note that the segments of a complex are always assumed to be non-perpendicular to the coordinate axes. From the finiteness of the number of the segments it follows that their inclination to the coordinate planes is bounded from below. Hence, Lemma 3 implies that a sufficiently small change of the beginning of a scheme will produce an arbitrarily small shift of the vertices of the scheme. These vertices will not disappear. These properties will be referred to as the continuous dependence of a stable generating system on its beginning. A finite number of stable generating schemes A_i depend in a uniformly continuous manner on the beginning, in the sense that for every $\varepsilon > 0$ there exists a $\delta > 0$ which is the same for all these schemes.

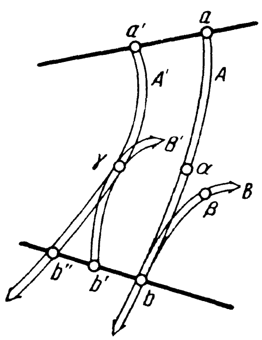


Figure 13. Relative to Lemma 4. The thick arrows represent the generating systems.

Lemma 4 (Figure 13). *Let us assume that the class of the point b does not contain a closed zigzag. Let A be a stable generating scheme which starts at a, and let B be a stable generating scheme with beginning at b, whose first direction is the same as that along which the scheme A leads into b. Then the points a and b have neighborhoods u_a and u_b such that if the scheme A' (this is a scheme that corresponds to A but its beginning is at $a' \in u_a$) passes through the point $b'' \in u_b$, then the scheme B' (which corresponds to B but leads away from $b' \in u_b$) has no points in common with A', provided $b' \neq b''$.*

Proof. Let us consider the set of points of the schemes A and B. Suppose that the shortest distance between two points is $\eta > 0$. We will pick for the points a and b neighborhoods u_a and u_b such that if $a' \in u_a$, $b' \in u_b$, then the points of the schemes A' and B' will be at a distance less than $\eta/3$ from the corresponding points of the schemes A and B. Such neighborhoods can be found in view of the remark relative to the Lemma 3. These are the neighborhoods sought.

Indeed, let a', b' and b'' be the points mentioned in the hypothesis of the lemma. Suppose that the point γ lies on A' and B' (Figure 13). As a point of A', it has a mate α on A. As a point of B', it has a mate β on B. We will prove that α and β coincide. Indeed, in the opposite case they would be at some distance not less than η from each other. But the point γ is at a distance less than $\eta/3$ from its mate β, and at a distance less than $\eta/3$ from α for the same reason. The obtained contradiction proves that $\alpha = \beta$. But this implies that the zigzag that connects b with β in B lies entirely in A; in the opposite case one could connect b with β $(= \alpha)$ by means of a zigzag through A in a different way. But the class of the point b, by the hypothesis of the lemma, contains no closed zigzags. The scheme A' is not longer than A. It contains γ, which corresponds to α, and it contains b'', which corresponds to b. This implies that A' contains a zigzag connecting b and γ of the same type as the zigzag $(b\alpha) \in A$. On the basis of similar arguments, the zigzags $(b\beta) = (b\alpha)$ and $(b'\gamma)$ are of the same type. The zigzags $b'\gamma$ and $b''\gamma$ must, therefore, also be of the same type. This, however, contradicts the nondegeneracy of the corresponding type of linear transformation because the points b' and b'' had been assumed to be distinct. This contradiction establishes Lemma 4.

In §8 we will make use of still another property of stable systems. We shall call it the property N. A scheme A which leads away from the point $a_0 \in K$, has the property N if the point α_0 lies on the segment $\Delta \subset K$, where it possesses neighborhoods* u_1 and u_2 (in the case when a_0 is an endpoint of K, α_0 has a one-sided neighborhood) such that for all points $a_0' \in u_1$ there exists a generating scheme $A'(a_0')$ of the same type and not longer than A, and for all points $a_0'' \in u_2$ there exists a generating scheme $A''(a_0'')$ of the same type not longer than A.

Examining Figure 12 one can understand that these types do not necessarily coincide, but may all three (type A, type A', and type A'') be different.

The following lemma is true:

Lemma 5. *Every stable generating scheme has the property* N.

Thus Lemma 5 can be deduced from Lemma 6 just as Lemma 2 can be deduced from Lemma 1.

Lemma 6. *Let* A *be a generating scheme that starts at* a_0 *in* K. *If each of the shortened schemes* A_i *of the scheme* A *has the property* N, *and the point* a_0 *is not a branching point of* K, *then the scheme* A *has the property* N.

The proof of Lemma 6 is analogous to the proof of Lemma 1.

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* That is, intervals which lie on Δ and have a_0 for a limit point.

We introduce now the concept of a generating scheme (system) of intervals.

For this purpose we consider a generating scheme of points of the interval u of the complex K. Suppose they are all of the same type (as, for example, those of the scheme A' of the points of the interval u_a in the definition of the property N). The set of corresponding points of these schemes form intervals in which the zigzags of one type map the interval u. The corresponding planes of these systems form layers. If the parallel layers do not intersect then we have a generating scheme of the interval u. It consists of the intervals of a scheme analogous to the ends which lie in the intersection of K with the layers of the scheme that are analogous to the planes. The interval of a scheme of rank 0 is u; the set of all planes of the first direction of the schemes of the points u is a layer of rank 1. It will lead from u and will lead to the intervals of rank 1. And so on. From the combinatorial viewpoint, a generating scheme of intervals is constructed in the same way as a generating scheme of points. In place of free ends we have here free intervals.

The following concept was not introduced for the schemes (systems) of points. An interval of a layer is the intersection of the layer with the coordinate axis that is perpendicular to the layer. The generating schemes of points u associate with each point u a point in each interval of the scheme and a plane in each layer. This defines a linear mapping u on each interval of the layer.

Applying the Lemmas 2, 3, and 5 to the tree D_n which has the inductive properties 1 and 2, we can establish that D_n has a generating scheme that leads from the point p_n, and from each point of the class p_n. This scheme (system) is stable, has the property N and depends continuously on its beginning. The scheme exists because D_n has the inductive property 2, and the class of the point p_n does not contain branching points in view of property 2. Thus, the lemmas are applicable to this scheme.

§6. Inductive preservation of generating schemes

In §4 it was shown how one can add to D_n a branch s_n, as small as we please, in such a way that D_{n+1} would have property 1. In order that D_{n+1} may have the inductive properties 2, 3, and 4, it is necessary that s_n be small enough. Having chosen the direction of the straight line l in accordance with §4, having selected $\varepsilon > 0$ sufficiently small, and then s_n in the ε-neighborhood p_n, as described in §4, we find that all four inductive properties hold on the constructed tree D_{n+1}.

In this section it will be proved that if D_n has the properties 1 and 2. and if the direction l has been chosen correctly, then there exists an $\varepsilon > 0$ such that if s_n is placed in the ε-neighborhood of p_n, then D_{n+1} possesses the inductive property 2.

In accordance with the property 2, the tree D_n has a finite set of types of canonical generating schemes. We shall transform these types somewhat. We shall try to obtain a finite number of generating schemes A_i which pass through p_n and which are such that for every $\delta > 0$ there exists an $\varepsilon > 0$ such that the planes of the canonic schemes of points lying outside a δ-neighborhood of the beginning a_i of the scheme A_1 will not intersect the ε-neighborhood of p_n.

Suppose that the existing canonic types do not possess this property. Since the number of types is finite, one of them must be nonregular. By this we mean that this type contains generating canonic schemes which have planes arbitrarily close to p_n if there is no scheme that passes through p_n. We select from the sequence of initial points of the indicated schemes, a sequence that converges to a, and we consider the set of limit points of the set of points of all these schemes. This set of limit points cannot be a generating scheme. But it contains p_n, and by adding to some of its points (their number is, obviously, finite) their generating schemes, we obtain the generating scheme of the point a. The added points are all distinct from each other and from those that existed before, because in the class of p_n there are no points of closed zigzags.

By Lemma 2, the obtained system is stable. Therefore, there must exist a neighborhood of the point a such that the generating schemes which start in this neighborhood must be schemes that correspond to this point, because of stability. Let us replace (in this neighborhood of a) the nonregular type of generating schemes by the schemes that correspond to a. The obtained types will be considered to be canonic. It is clear that the remainder of the canonic nonregular type is regular. This can be proved easily by making use of the linearity of the corresponding mappings.

Having performed this operation with all the old nonregular types, we obtain new canonic types; we shall call them simply canonic types. A finite number of the points a_i have canonic schemes passing through p_n. All nonregular types are now in the intervals of stability of these schemes A_i. From the linearity of the mapping of the neighborhood of a_i into the neighborhood of p_n with the aid of the corresponding zigzags of the canonic schemes, there follows the following assertion.

For every $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that the* ε*-neighborhoods* *of the point p_n can be intersected by the planes of only those new canonic schemes whose beginnings lie in a δ-neighborhood of the points a_i, and which correspond to the A_i in the sense of stability.*

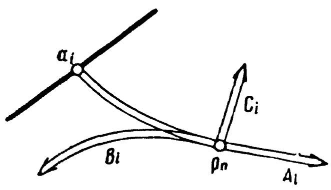

Figure 14. The thick arrows represent generating schemes.

Let A_i be a canonic generating scheme that leads away from a_i and passes through p_n. We shall consider the following generating schemes (Figure 14):

B_i, the scheme that leads away from p_n in the same direction along which A_i arrived at p_n.

C_i, the generating scheme, leading away from p_n, whose first direction is different from the ones along which A_i leaves p_n and arrives at p_n, and from the first direction of B_i. (In case $p_n = a_i$, the scheme B_i is not defined, and we do not consider it.)

All these schemes pass through p_n, and they are, therefore, stable.

According to the inductive condition 1, the constructed generating schemes have no points in common besides p_n, and the B_i and A_i satisfy condition 4.

Let us consider the set of all the points of all three schemes. Let the positive number η be the least distance between any two points of this set. Applying Lemma 4 to A_i and B_i, we find a δ-neighborhood of the point a_i, and an ε-neighborhood of the point p_n such that A_i and B_i' will not intersect if their beginnings lie in the indicated neighborhoods (for the definitions of A' and B' see Lemma 4). Applying Lemma 3 to the schemes A_i, B_i, C_i, we find a $\delta_2 > 0$ and an $\varepsilon_2 > 0$ such that all the points of A_i', B_i', C_i' will be at a distance greater than $\eta/3$ from their corresponding points of A_i, B_i, C_i provided that the beginning of A_i' lies in a δ_2-neighborhood of a_i, and the beginnings of the remaining schemes in an ε_2-neighborhood of p_n. Here C_i' is a scheme of the same type as C_i but shorter.

Let us choose a positive number δ less than δ_1 and δ_2. For this we find an $\varepsilon_3 > 0$ such that the ε_3-neighborhood of p_n is intersected by the planes of only those canonic generating schemes whose beginnings lie in δ-neighborhoods of the points a_i and whose first directions are the same as those of A_i. We choose a positive number ε less than ε_1, ε_2, and ε_3. From the finiteness of the number of types A_i it follows that all the numbers ε and δ can be chosen uniformly for all i. Consequently, we can obtain a system of δ-neighborhoods of the points a_i, and ε-neighborhoods of the p_n such that the following statements are true.

A) The ε-neighborhood of p_n is intersected by the planes of only those canonic schemes whose beginnings lie in δ-neighborhoods of the points a_i, and which correspond to A_i.

B) The schemes A_i' and B_i' do not intersect if their beginnings lie in the indicated δ- and ε-neighborhoods.

C) In the transition from A_i, B_i, C_i to A_i', B_i', C_i' the points of these schemes will be shifted over distances less than $\eta / 3$ provided the beginnings remain within the indicated neighborhoods.

Recalling the meaning of the positive number η, we see that if the beginnings lie in the indicated neighborhoods, then the schemes B_i', C_i' cannot have any points in common besides the beginning. The same thing is true for C_i' and A_i', B_i' and A_i'.

Let us inscribe a parallelepiped P in the obtained neighborhood of p_n. The edges of the parallelepiped are to be parallel to the coordinate axes, its center is to be at p_n, and one of its diagonals is to lie on q_n. Inside P we attach to q_n at p_n a segment $2s_n$ in the direction l (see §3) (Figure 15).

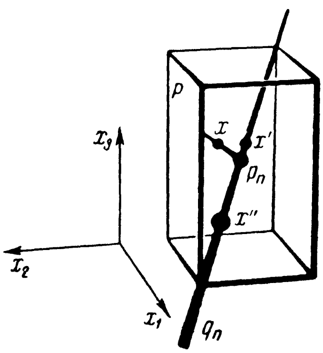

Figure 15. The attaching of s_n.

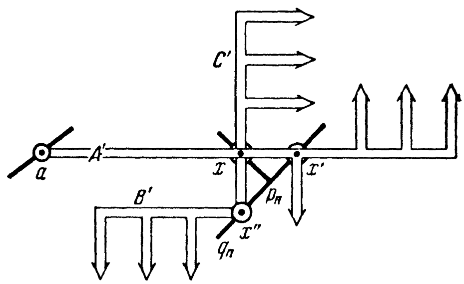

Figure 16. Generating scheme leading away from the point a on D_{n+1}.

The length of the segment s_n will be restricted from above. In order to preserve property 2 on D_{n+1}, it is sufficient that $2s_n \subset P$. We shall prove this.

If the planes of the canonic generating scheme D_n, which leads away from some point of D_n, do not intersect $2s_n$, then the scheme will remain to be a generating scheme also on D_{n+1}. In particular, this is known to be the case for all points that lie outside the δ-neighborhoods of the points a_i. From the property A) it follows that it is sufficient to construct a generating scheme leading away from each point of the δ-neighborhood of a_i in the first direction of A_i. Let A be such a point. The canonic scheme A_i' which leads away from a on D_n in the first direction of the scheme A_i, corresponds to A_i because the δ-neighborhood is smaller than the interval of stability.

Suppose that A intersects $2s_n$ at the point x. Then A_i' intersects q_n in some point x'; $x \neq x'$ if $a = a_i$.* In the sequel we will assume that $a \neq a_i$. Let us pass a plane through x'. The first direction of this plane is the same as that of C_i. Suppose that x'' is the point of intersection of this plane with q_n. From the choice of the direction of $2s_n$ it follows that $x'' \neq x'$ and x.

Let us construct (Figure 16) generating schemes C_i' and B_i' leading away from the points x''. From the properties B) and C) of the ε- and δ-neighborhoods it follows that the schemes A_i' and B_i', as well as the schemes A_i' and C_i' have no points in common, while B_i' and C_i' have only the beginning in common. It is easy to see that the planes A_i', B_i', C_i', that do not pass through x, x', x'', cannot intersect P.

All the thus far considered generating schemes led away from D_n. With their aid one can construct, however, schemes which will lead away from an a on D_{n+1}. The scheme A_i' does not lead to D_{n+1} only because the point x is not free on it. Let us select a direction at this point for the first plane of the scheme C_i. The obtained plane intersects D_{n+1} (in addition to the point x) also at the point x'' and at points of the first rank of the scheme C_i'. From the points of the first rank we leave along directions, along which we pass to C_i'. B_i' leads away from the point x'' on D. Since these schemes do not intersect, except at the point x'', because they cannot have any points in common with A_i, and since the planes of the schemes A_i', B_i', C_i' do not intersect P (except for the four planes which are here being considered and pass through x, x', x''), we obtain a generating scheme that leads away from a to D_{n+1}. In case $a_i = p_n$, and $a \in 2s_n$, one does not have to construct x''; the scheme C_i' is constructed at the point x' (Figure 17). The proof is analogous to the preceding one.

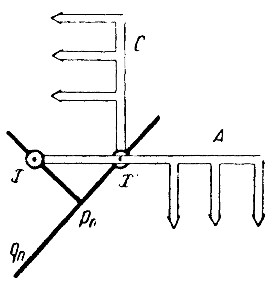

Figure 17. Generating scheme that leads from x on D_{n+1}.

The proof of the inductive fulfillment of the property 2 on D_{n+1} under the conditions of the fulfillment of the properties 1 and 2 on D_n, will be finished as soon as we give the finite number of types of generating schemes. But we have actually done

* In case $a = a_i$, the scheme A_i remains a generating scheme on D_{n+1}.

this in the construction of the generating schemes leading away from a. Indeed, it is easy to see that the set of types of schemes, which are here used (schemes A_i', B_i', and C_i') is finite, because they are not longer than the schemes A_i, B_i, C_i which are the schemes of the canonic type on D_n.

§7. Inductive preservation of the decomposition of functions

This section contains the construction of the representation of a function (defined on a finite tree D_n) in the form of the sum of functions of the coordinates.

Lemma 7. *Let A be a scheme that leads away from a point a_0 of a complex K, and let $f(x)$ be a function, defined on K, which differs from zero at the point a_0 only. Then there exists functions of the coordinates x_k of the point x such that for every point $x \in K$*

$$f(x) = \sum_{k=1}^{3} f_k(x_k), \qquad (*)$$

where the functions $f_k(x)$ differ from zero only at those points of the kth axis which are the intersections of this axis with the planes of the scheme A.

Proof. Let us assume that $f_k^0(x_k) \equiv 0$. If we substitute f_k^0 into the right-hand side of equation (*) then this equation will fail to hold only at the point of rank 0 of the scheme A. We will call the function $f_k^0(x_n)$ the zeroth approximation to $f_k(x_k)$. The function of the nth approximation, $f_k^n(x_k)$, will be constructed in such a way that the following conditions hold.

1) If the function $f_k^n(x_k)$ is substituted for $f_k(x_k)$ in the equation (*), then this equation will fail to hold only at the points of rank n of the scheme A.

2) $f_k^n(x_k) = f_k^{n-1}(x_k)$ $(n = 1, 2, \dots)$, if the point x_k of the kth axis does not lie on planes of rank n of the scheme A.

The functions of the zeroth approximation possess the property 1), and if the rank of the scheme A is N, then the $f_k^{N+1}(x_k)$ will satisfy, obviously, all the requirements of Lemma 7. If the $f_k^{n-1}(x_k)$ are constructed so that the conditions 1) and 2) hold, then we set

$$f^{n-1}(x) = \sum_{k=1}^{3} f_k^{n-1}(x_k).$$

The expression $f(x) - f^{n-1}(x) = \Delta^n(x)$, the nth disjoint, is different from zero at the points of rank $n - 1$ of the scheme A. Let a be such a point, and suppose that the plane π, which leads away from this point, intersects the kth axis at the point $x_k(a)$. From the definition of the generating system it follows that all the $x_k(a)$, that correspond to different a and n, are distinct. Introducing corrections $\Delta_k^n(x_k) = \Delta^n(a)$ for $f_k^{n-1}(x_k)$ at the points $x_k(a)$, we set $f_k^n(x_k) = f_k^{n-1}(x_k) + \Delta_k^n(x_k)$. It is obvious that $f_k^n(x_k)$ has the properties 1), 2). Hence, one can construct all the $f_k^{n+1}(x_k)$. This completes the proof of Lemma 7.

Lemma 8. *Let A be a scheme which leads away from the interval s of the complex K, and let $f(x)$ be a continuous function, defined on K, and differing from zero on s only. Then there exist continuous functions $f_k(x)$ of the coordinates of the point x such that for every point $x \in K$*

$$f(x) = \sum_{k=1}^{3} f_k(x_k),$$

where the functions $f_k(x_k)$ are different from zero only on the intervals of the layers of the scheme A.

The proof of this assertion is analogous to the proof of Lemma 7. All points and planes are replaced by intervals and layers, while the functions which differ from zero at separate points are replaced by continuous functions differing from zero only on separate nonintersecting intervals at whose ends they are zero. In particular, the disjoints and corrections will be such functions.

Lemma 9. *The assertions of Lemmas 7 and 8 are true for double schemes.*

Proof. The proof of this lemma is again accomplished with the aid of the distribution of corrections. At the points (intervals) from which two planes (layers) issue, one may ignore one of them, obtain a simple system and make use of the Lemma 7 (8). But then the corrections and disjoints of all ranks will be equal. One can decrease the size of the corrections if one makes use of both issuing planes (layers) for the "distribution of the corrections along two directions".

Suppose, for example, that the planes π_1 and π_2 of the directions x_1 and x_2, respectively, lead away from the point a. In order that the equation (*) may hold at the point a_0, one may set

$$\Delta_1^1(x_1) = \gamma_1 \Delta^1(x), \quad f_1^1(x_1) = f_1^0(x_1) + \Delta_1^1(x_1),$$

$$\Delta_2^1(x_2) = \gamma_2 \Delta^1(x), \quad f_2^1(x_1) = f_2^0(x_2) + \Delta_2^1(x_2),$$

where, as before $f_k^0(x_k) \equiv 0$, $\Delta^1(x) = f(x) - \sum_{k=1}^{3} f_k^0(x_k)$ and where $\gamma_1, \gamma_2 > 0$,

$\gamma_1 + \gamma_2 = 1$. Then the equation (*) will fail to be satisfied at all ends of rank 1, and one has to introduce the correction at a greater number of points than one would have had to if one had ignored π_2 by assuming that $\gamma_2 = 0$.

In the final construction of the functions $f_k(x_k)$ in §9, use will be made of the distribution along two directions.

Lemma 10. *Suppose that we are given on a segment-like complex K two continuous functions: an "old" one,*

$$\bar{f}(x) = \sum_{k=1}^{3} \bar{f}_k(x_k),$$

where the $\bar{f}_k(x_k)$ are continuous functions of the coordinates x_k of the point $x \in K$, and a "new" one $f(x)$, which differs from the old one only on the interval s that possesses on K a generating scheme A (simple or double). Then one can find "corrections for $\bar{f}_k$" which are continuous functions $g_k(x_k)$, differing from zero only on the intervals of the layers of the scheme A, and which are such that if one writes $f_k(x_k) = \bar{f}_k(x_k) + g_k(x_k)$, then on the entire complex K

$$f(x) = \sum_{k=1}^{3} f_k(x_k).$$

Lemma 10 is a direct consequence of the Lemmas 8 and 9 if one introduces the function $g(x) = f(x) - \bar{f}(x)$. The process of the distribution of the corrections along two directions, which leads to the construction of the $g_k(x_k)$ $(\sum_{k=1}^{3} g_k(x_k) = g(x))$, determines the disjoints $\Delta^n(x)$, the corrections $\Delta_k^n(x_k) = \gamma_k(x)\,\Delta^n(x)$, and the approximations $g_k^n(x_k) = g_k^{n-1}(x_k) + \Delta_k^n(x_k)$. It is clear that one may consider the functions $f_k^n(x_k) = \bar{f}_k(x_k) + g_k^n(x_k)$ as approximations to the $f_k(x_k)$; the disjoints $f(x) - \sum_{k=1}^{3} f_k^n(x_k)$ and corrections $f_i^n(x_i) - f_i^{n-1}(x_i)$ will hereby be the same. The construction of the $f_k^n(x_k)$ $(k = 1, 2, 3;\ n = 0, 1, 2, \dots, N + 1)$, which was described above, will be called the distribution of corrections.

Lemma 11. *For the preservation on D_{n+1} of the inductive property 3, it is sufficient that the interval $2s_n$ have a generating scheme on $D_n \cup 2s_n$. The expansion of $f(x)$ as a sum of functions f_k^{n+1} of the coordinates can be accomplished through the introduction of corrections for f_k^n with the aid of the distribution of corrections along two directions determined, in general, by the double generating scheme of $2s_n$.*

Proof. Suppose that on $D_n \cup 2s_n$, the interval $2s_n$ has a generating scheme. On D_n, every continuous function can be expressed as the sum of functions of coordinates (inductive requirement 3_n). We select for the old function $f^n(x) = \sum_{k=1}^{3} f_k^n(x_k)$, and for the new function, $f(x)$ on D_{n+1}. On $D_n \cup 2s_n$ we define this function so that the difference between it and the old function on $2s_n$ is an even function relative to the midpoint of $2s_n$. Then we will have the conditions of Lemma 10, from which follows the possibility of the representation of $f(x)$ on D_{n+1} as the sum of functions of the coordinates by the method of the distribution of the corrections along two directions. If each correction depends continuously on the expanded function (and this can, obviously, be obtained from the conditions of Lemmas 7-11), then the expansion $f(x) = \sum_{k=1}^{3} f_k^{n+1}(x_k)$ depends continuously on f. In §7 every correction depends continuously on the expanded function.

If the branch s_n is constructed as indicated in §§3-5, then the requirements 1_{n+1} and 2_{n+1} will be satisfied on D_{n+1}. The last requirement means that there exists on D_{n+1} a finite number of canonic generating schemes of intervals. For this it is only necessary that (§5) the direction of s_n be chosen correctly and that the branch s_n lie in a sufficiently small neighborhood P of the point p_n.

Lemma 12. *Suppose that the conditions* 1_n *and* 2_n *are fulfilled on* D_n. *If* s_n *lies in a small enough neighborhood* $P' \subset P$ *of the point* p_n, *then* $2s_n$ *has a generating scheme on* $D_n \cup 2s_n$.

Proof. Let us consider the above constructed canonic generating schemes of the points $2s_n'$ on D_{n+1} with a given first direction (Figure 17). When x changes on $2s_n'$, then x' runs through a one-sided neighborhood u of the point p_n on q_n. Because of the stability of the schemes A and C, there exists a semineighborhood u on the same side of p_n for whose points all schemes A' and all schemes C' will be of the same type (Lemma 5).

The points and parallel planes A and C do not intersect in pairs (excepting at the point p_n). From the continuous dependence of A' and C' on the initial point x' it follows that if x' changes in a sufficiently small neighborhood of the point p_n, then the points and planes of the schemes A' and C' will be as close as we please to the corresponding elements of the schemes A and C. Let $\eta > 0$ be the least of the distances between any plane π of one of the schemes A, C and a point (not on π) of one of these schemes. Let $\varepsilon > 0$ be the radius of a neighborhood of the point p_n such that the planes and points of the schemes A and C are shifted by not more than $\eta/3$ when the point x' varies over the ε-neighborhood of p_n. Then the intersection $\tilde{u}$ with the ε-neighborhood of p_n will yield a semineighborhood of the point p_n which is an interval u having on D_{n+1} nonintersecting generating schemes (see §5) A^* and C^*.

This follows from the fact that all schemes A' and C' are of the same type; any two parallel layers of these schemes will, obviously, not be separated by a distance greater than $\eta/3$. If we now place the segment $2s_n$ in a small enough neighborhood P_1 of the point p_n (namely such that x' falls into $\hat{u}$) then the interval $2s_n'$ will have a generating scheme on D_{n+1} whose first direction will coincide with the first direction of A.

From the Lemmas 11 and 12 it follows that *for the preservation of the inductive property* 3 *on* D_{n+1} *it is sufficient that the segment* s_n *be small and have a properly chosen direction* (§§3-5).

In §7 use is made of a generalization of the Lemma 12.

By the N-characteristic χ_N of a generating scheme A_u of the interval u on K we shall mean the set of directions of the generating layers of the intervals of rank less than N referred to these intervals. The N-characteristic χ_N and K determine uniquely the elements of the scheme A_u whose rank does not exceed N.*

Lemma 13. *Suppose that the conditions* 1_n *and* 2_n *are fulfilled on* D_n. *For every* $N > 0$, *there exist a neighborhood* $P(N)$ *of the point* p_n *and generating schemes of intervals* $u \subset P(N)$ *such that*

1) *Among them there exist schemes* $A_u^{\chi_N}$ *with any* N*-characteristic.*

2) *The intervals of the schemes* $A_u^{\overset{1}{\chi}N}$ *and* $A_u^{\overset{2}{\chi}N}$, *different from* u, *do not intersect if the first directions of these schemes are distinct.*

3) *If the intervals* $u_1 \in P(N)$, $u_2 \subseteq P(N)$ *do not intersect, then none of the intervals of the schemes* $A_{u_1}^{\chi_N}$ *and* $A_{u_2}^{\chi_N}$ *will intersect.*

The proof of Lemma 13 is analogous to the proof of Lemma 12, and is left to the reader.

Up to now our constructions have not depended on whether the function f belongs to the class F which is mentioned in the inductive lemma. In §9 the expansion constructed here will depend on F. This will not destroy the possibility of expanding any function on D_n into the sum of functions of the coordinates. We can, obviously, without loss of generality, assume that F is a compact. It is easy to see that within the limits of F, the continuous dependence of f_k^n on f is uniform.

* In the N-characteristic one can indicate the directions of the layers that lead away from the intervals which are not in the scheme A_u, because this scheme may terminate earlier with a free end.

§8. Arithmetic lemma

In this section there are proved two lemmas with whose aid there will be obtained, in the next section, corrections along two different directions.

Lemma 14. *Let*

$$a+b+c=d, \tag{1}$$

where

$$|a|, \quad |b|, \quad |c|<3+\theta, \tag{2}$$

$$|d| \leqslant 1. \tag{3}$$

Let

$$a'=a+\Delta a, \tag{4}$$

where

$$|\Delta a|<1+\varepsilon, \tag{5}$$

$$0<\varepsilon<\theta<1. \tag{6}$$

Then one can determine numbers $\Delta b(a, b, c, \Delta a)$ *and* $\Delta c(a, b, c, \Delta a)$ *such that if*

$$b'=b+\Delta b, \quad c'=c+\Delta c \tag{7}$$

then

$$|b'|, \quad |c'|<3+\theta+\varepsilon, \tag{8}$$

$$a'+b'+c'=d \tag{9}$$

$$|\Delta b|, \quad |\Delta c| \leqslant \max\left(\left| |\Delta a|-\frac{\varepsilon^2}{30}\right|, \varepsilon\right) \tag{10}$$

and, such that

$$\left.\begin{array}{l}\text{the dependence of } \Delta b \text{ and } \Delta c \text{ on } a, b, c, \Delta a, \text{ which vary} \\ \text{within the restrictions (1) to (6), be continuous and that} \\ \text{as } \Delta a \to 0, \ \Delta b \text{ and } \Delta c \text{ will tend to zero.}\end{array}\right\} \tag{11}$$

Proof. We shall prove first that under the conditions of the lemma

$$|b+c|<4+\theta. \tag{12}$$

Indeed, from (1) it follows that $b+c=d-a$. Therefore, $|b+c| \leqslant |d|+|a|$. But since according to (2) and (3), $|a|<3+\theta$, $|d|<1$, it follows that $|b+c|<4+\theta$. From (12) and (2) we obtain

$$2(3+\theta) \pm(b+c)>2+\theta. \tag{13}$$

In order to satisfy the requirements of the lemma, we define Δb and Δc as

$$\begin{aligned} \Delta b &= -\gamma_b \Delta a, \quad 0<\gamma_b<1, \\ \Delta c &= -\gamma_c \Delta a, \quad 0<\gamma_c<1. \end{aligned} \tag{14}$$

If $\gamma_b + \gamma_c = 1$, then (9) will be fulfilled.

If here γ_b and γ_c depend continuously on $a, b, c, \Delta a \neq 0$, then (11) is satisfied. In order to select γ_b and γ_c so that the inequalities (8) will not be violated, we introduce

$$\begin{aligned} \lambda_b^- &= 3+\theta-b+\frac{\varepsilon}{2}, \quad \lambda_c^- = 3+\theta-c+\frac{\varepsilon}{2}, \\ \lambda_b^+ &= 3+\theta+b+\frac{\varepsilon}{2}, \quad \lambda_c^+ = 3+\theta+c+\frac{\varepsilon}{2}. \end{aligned} \tag{15}$$

These numbers, which are positive because of (2), give the leeway which one has for the introduction of the corrections Δb and Δc; thus, for example, λ_b^- shows how much one may add to b in order that the sum b' may not exceed $3 + \theta + \varepsilon/2$ (see (8)).

The inequality (13) shows the correction Δa, which does not exceed 2 in absolute value, can be made to satisfy (8) by selecting γ_b and γ_c in (14) between 0 and 1. Namely, if $\Delta a > 0$, we set

$$\gamma_b = \frac{\lambda_b^+}{\lambda_b^+ + \lambda_c^+}, \quad \gamma_c = \frac{\lambda_c^+}{\lambda_b^+ + \lambda_c^+}, \tag{16a}$$

and if $\Delta a < 0$, we let

$$\gamma_b = \frac{\lambda_b^-}{\lambda_b^- + \lambda_c^-}, \quad \gamma_c = \frac{\lambda_c^-}{\lambda_b^- + \lambda_c^-}. \tag{16b}$$

We shall prove that (1)-(7), (14), (15), (16a), and (16b) imply (8), (9), (10), (11). Indeed, (9) is satisfied because of the obvious equation $\gamma_b + \gamma_c = 1$. From (12), (13), and (15) we obtain

$$2 < \lambda_b^\pm + \lambda_c^\pm = 2\left(3+\theta+\frac{\varepsilon}{2}\right) \pm (b+c) < 15, \tag{17}$$

and therefore, γ_b and γ_c will depend continuously on $a, b, c, \Delta a$ when $\Delta a \neq 0$. Since $0 < \gamma_b$, $\gamma_c < 1$, the condition (11) is satisfied. From (5), (6), and (7) it follows that

$$\frac{|\Delta a|}{\lambda_b^\pm + \lambda_c^\pm} < \frac{1+\varepsilon}{2} < 1.$$

Therefore, $|\Delta b| < \lambda_b^\pm$, $|\Delta c| < \lambda_c^\pm$. But from (15) it follows that

$$|b \mp \lambda_b^\pm| < 3+\theta+\varepsilon, \qquad |c \mp \lambda_c^\pm| < 3+\theta+\varepsilon,$$

and because of (7), (14), (16a), and (16b), $|b'| < 3 + \theta + \varepsilon$, $|c'| < 3 + \theta + \varepsilon$, i.e. (8) is fulfilled. It remains to prove that (10) holds. In case $|\Delta a| \leqslant \varepsilon$, (10) is, obviously, a consequence of the relations $0 < \gamma_b < 1$, $0 < \gamma_c < 1$. From (15) and (2) it follows that $\lambda_{b,c}^\pm > \varepsilon/2$. Hence, in view of (17),

$\gamma_{b,c} > \varepsilon/30$. From this we have in accordance with (14) that $|\Delta b|, |\Delta c| > |\Delta a|\, \varepsilon/30$. Therefore, in case $|\Delta a| \geqslant \varepsilon$ it follows that $|\Delta b| > \varepsilon^2/30$, $|\Delta c| > \varepsilon^2/30$. But since (see (14) and (16)) $|\Delta b| + |\Delta c| = |\Delta a|$, it now follows that $|\Delta b| < |\Delta a| - \varepsilon^2/30$, $|\Delta c| < |\Delta a| - \varepsilon^2/30$, namely the condition (10) and Lemma 14 have been proved.

Lemma 15. *Let*

$$a + b + c = d \tag{1}$$

$$|a|,\quad |b|,\quad |c| < 3 + \theta, \tag{2}$$

$$|d| < 1 + \varepsilon. \tag{3}$$

Let

$$d' = d + \Delta d, \tag{4}$$

where

$$|\Delta d| < 1 + \varepsilon, \tag{5}$$

$$0 < \theta \leqslant 1, \quad 0 < \varepsilon < 1. \tag{6}$$

Then one can determine the numbers $\Delta a(a, b, c, \Delta d)$ *and* $\Delta b(a, b, c, \Delta d)$ *so that if*

$$a' = a + \Delta a, \quad b' = b + \Delta b \tag{7}$$

then

$$a' + b' + c = d', \tag{8}$$

$$|a'| < 3 + \theta + \varepsilon, \quad |b'| < 3 + \theta + \varepsilon, \tag{9}$$

$$|a - \Delta b| < 3 + \theta + \frac{\varepsilon}{2} \tag{10}$$

and that

$$\left.\begin{array}{l}\text{the dependence of } \Delta a \text{ and } \Delta b \text{ on } a, b, c, \text{ and } \Delta d, \text{ which} \\ \text{vary within the given (see (1)-(6)) limits, will be con-} \\ \text{tinuous, and if } \Delta d \to 0 \text{ then } \Delta a \text{ and } \Delta b \text{ will go to zero.}\end{array}\right\} \tag{11}$$

Proof. For the fulfillment of the inequalities (9) it is sufficient that

$$\begin{array}{l} 0 \leqslant \Delta a < \lambda_a^+ \quad \text{or} \quad -\lambda_a^- < \Delta a \leqslant 0, \\ 0 \leqslant \Delta b < \lambda_{b1}^+ \quad \text{or} \quad -\lambda_{b1}^- < \Delta b \leqslant 0, \end{array} \tag{12}$$

where

$$\begin{array}{ll} \lambda_a^+ = 3 + \theta + \varepsilon - a, & \lambda_a^- = 3 + \theta + \varepsilon + a, \\ \lambda_{b1}^+ = 3 + \theta + \varepsilon - b, & \lambda_{b1}^- = 3 + \theta + \varepsilon + b, \end{array} \tag{13}$$

since a and b satisfy relation (2).

In order that (10) be satisfied, it is sufficient that

$$0 \leqslant \Delta b < \lambda_{b2}^+ \quad \text{or} \quad -\lambda_{b2}^- < \Delta b \leqslant 0, \tag{14}$$

where

$$\lambda_{b2}^{+}=3+\theta+\frac{\varepsilon}{2}+a, \quad \lambda_{b2}^{-}=3+\theta+\frac{\varepsilon}{2}-a, \tag{15}$$

again because of (2).

Setting now

$$\lambda_{b}^{+}=\min(\lambda_{b1}^{+}, \lambda_{b2}^{+}), \quad \lambda_{b}^{-}=\min(\lambda_{b1}^{-}, \lambda_{b2}^{-}), \tag{16}$$

we find that

$$\lambda_{b}^{+}+\lambda_{a}^{+}>2, \quad \lambda_{b}^{-}+\lambda_{a}^{-}>2. \tag{17}$$

Indeed, by (1) we have, $a+d=d-c$. Therefore, $|a+b| \leqslant |d|+|c|$ and, by (2) and (3),

$$|a+b|<4+\theta+\varepsilon. \tag{18}$$

But because of (13), $\lambda_{a}^{+}+\lambda_{b1}^{+}=6+2\theta+2\varepsilon-(a+b)$. Hence, it follows from (18) that $\lambda_{a}^{+}+\lambda_{b1}^{+}>2$. At the same time we have, in view of (13) and (15), that $\lambda_{a}^{+}+\lambda_{b2}^{+}=6+2\theta+2\varepsilon>2$. In accordance with (16), the first inequality of (17) has been established; the second one can be proved to be valid in a similar way.

Now we set

$$\Delta a=\gamma_{a}\Delta d, \quad \Delta b=\gamma_{b}\Delta d, \tag{19}$$

where if $\Delta d>0$,

$$\gamma_{a}=\frac{\lambda_{a}^{+}}{\lambda_{a}^{+}+\lambda_{b}^{+}}, \quad \gamma_{b}=\frac{\lambda_{b}^{+}}{\lambda_{a}^{+}+\lambda_{b}^{+}} \tag{20a}$$

and if $\Delta d<0$,

$$\gamma_{a}=\frac{\lambda_{a}^{-}}{\lambda_{a}^{-}+\lambda_{b}^{-}}, \quad \gamma_{b}=\frac{\lambda_{b}^{-}}{\lambda_{a}^{-}+\lambda_{b}^{-}}. \tag{20b}$$

We shall prove that (1)-(7), (13), (15), (16), (19), (20a), and (20b) imply (8), (9), (10), and (11).

Indeed, from (20) we obviously obtain $\gamma_{a}+\gamma_{b}=1$, which implies (8) in view of (19), (1), (4), and (7). From (2), (13), (15), and (16) it follows that every λ is positive, and, hence, that $0<\gamma_{a}<1$, $0<\gamma_{b}<1$. Since, if $\Delta d \neq 0$, the γ_{a} and γ_{b} depend continuously on a, b, c, and Δd (see (20a) and (20b)), it now follows that (11) must be fulfilled because of (19).

Finally, from (17), (3), and (16) we obtain

$$\frac{\Delta d}{\lambda_{a}^{+}+\lambda_{b}^{+}}<\frac{1+\varepsilon}{2}<1, \quad \frac{-\Delta d}{\lambda_{a}^{-}+\lambda_{b}^{-}}<\frac{1+\varepsilon}{2}<1. \tag{21}$$

Taking into account the fact that λ is positive, we obtain with the aid of (20a), (20b), (19) and (21) the inequalities

$$0 \leqslant \Delta a < \lambda_a^+ \quad \text{or} \quad -\lambda_a^- < \Delta a \leqslant 0,$$
$$0 \leqslant \Delta b < \lambda_b^+ \quad \text{or} \quad -\lambda_b^- < \Delta b \leqslant 0,$$

These inequalities and (16) imply the relations (12) and (14). From (12) follows (9), and (14) implies the inequality (10). This completes the proof of Lemma 15.

§9. Inductive preservation of property 4

In this section it will be shown how one must distribute the corrections in the method of §7 in order to fulfil the inductive requirement 4_{n+1}.

In §3 we introduced the numbers n_r. The oscillation of any function f of the considered class F on any component of the complement of Δ_n in Ξ does not exceed $1/r^2$ provided $n \geqslant n_r$. In particular, this will be the case on each branch σ_n if $n \geqslant n_r$.

We will denote by $f^n(\xi)$ the function defined on Δ_n which coincides on Δ_n with $f \in F$, and also its continuous extension (over any Δ_m $(m > n)$ and on the entire Ξ) which is constant on each component of the complement of Δ_n in Ξ. That such an extension exists, and is unique, follows directly from the fact that the intersection of Δ_n with the closure of each component $\Xi \setminus \Delta_n$ consists of one point. The function which corresponds to $f^n(\xi)$ on D_m we will denote by $f^n(x)$ on X. Let us introduce the function

$$g^m(x) = f^m(x) - f^{n_r}(x) \tag{1}$$

$(n_r < m \leqslant n_{r+1})$. On D_{n_r} this function is zero, depends continuously on $f \in F$, and does not exceed $1/r^2$ anywhere in view of the definition of r and $f^m(x)$.

Let $n_r \leqslant n < n_{r+1}$. Suppose that D_n and $f_k^n(x_k)$ are determined so that the requirements 1_n, 2_n, 3_n, and 4_n are satisfied. Then (for $n = n_r$ this is trivial)

$$| f_k^n(x_k) - f_k^{n_r}(x_k) | < \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right)\frac{1}{r^2}. \tag{2}$$

Our problem consists of selecting s_n and $f_k^{n+1}(x_k)$ so that the requirements 3_{n+1} and 4_{n+1} will be fulfilled.

From here on, till the end of this section, r will be kept fixed. In order to shorten the formulas in all estimates, the factor $1/r^2$ will be omitted. Thus, the inequality (2) will be written now in the form

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| < 3 + \frac{n - n_r}{n_{r+1} - n_r}. \qquad (2')$$

This can be considered as a temporary change of the scale of the f-axis, or one can suppose that we are confining ourselves to the case $r = 1$, $1/r^2 = 1$, because the remaining cases can be treated in an analogous manner.

Thus, let us assume that on D_n the requirements 1_n, 2_n, 3_n, and 4_n are satisfied. Then on D_n

$$g^n(x) = \sum_{k=1}^{3} g_k^n(x_k), \qquad (3)$$

where $g_k^n(x_k) = f_k^n(x_k) - f_k^{n_r}(x_k)$ when $n > n_r$, and when $n = n_r$, $g_k^n(x_k) = 0$, $g^n(x) = 0$. As usual, the x_k are the coordinates of the point x. In (3) $x \in D_n$. The fulfillment of the requirement 4_n on D_n means that

$$|g_k^n(x_k)| < 3 + \theta_n, \qquad (4)$$

where we have introduced the notation

$$\theta_n = \frac{n - n_r}{n_{r+1} - n_r}. \qquad (5)$$

We will construct D_{n+1} in accordance with §7, and will select functions $g_k^{n+1}(x_k)$, which depend continuously on $f \in F$, in such a way that if $x \in D_{n+1}$

$$\sum_{k=1}^{3} g_k^{n+1}(x_k) = g^{n+1}(x), \qquad (6)$$

$$|g_k^{n+1}(x_k)| < 3 + \theta_{n+1}. \qquad (7)$$

Here, $n_r < n + 1 \leqslant n_{r+1}$, and one has to assume that

$$f_k^{n+1}(x_k) = f_k^{n_r}(x_k) + g_k^{n+1}(x_k), \qquad (8)$$

in order to prove 3_{n+1} and 4_{n+1}.

When n increases from n_r to n_{r+1}, then θ_n increases from 0 to 1, and when n increases by 1, θ_n increases each time by $1/(n_{r+1} - n_r)$. We choose ε, $0 < \varepsilon < 1/(n_{r+1} - n_r)$. Then $\theta_n + \varepsilon < \theta_{n+1}$. This will be kept fixed in the remainder of this section.

Construction of $2s_n$. On D_n there exists a point p_n where s_n is to be attached.

Let us consider the rays l' and l'' (Figure 18), into which the point

p_n divides the line containing q_n. When the direction s_n has been chosen, then the three coordinates which pass through $2s_n$ will intersect these rays. Let us now select the direction $2s_n$ so that one of these rays l' (it will be called the principal ray) will intersect the planes of one direction; this direction will be called the principal direction. The planes of the remaining two directions will intersect the ray l'' (it will be called the minor direction). One of these planes is chosen arbitrarily and is called the minor plane. Finally, this entire operation can be performed by not picking s_n from the forbidden directions of §4, which is now assumed. The direction s_n has been chosen.

The following assertions are true.

A. From every sufficiently small semineighborhood u_{pr} of the point p_n on the principal ray, one can start on D_n a double scheme A of the interval u_{pr} so that two layers will lead away from the intervals of ranks $1, 2, \ldots, N$, where N is taken equal to $[30/\varepsilon^2] + 1$ (in order to have $N\varepsilon^2/30 > 1$), and such that among the directions of the layers of rank 1 there is no principal direction.

From every sufficiently small semineighborhood u_{m} of the point p_n on a minor ray one can start on D_n a double scheme B of the interval u_{m} so that two layers will lead away from the intervals of ranks $1, 2, \ldots, N$, and that the first direction is the principal one. The symbol N has the same meaning here as in the preceding paragraph. The scheme C with the same N-characteristic can be started from the semineighborhood u_{pr} if this neighborhood is small enough. Finally, if the interval u_{m} is sufficiently small, then, on D_n, one can start from this neighborhood a double generating scheme D whose first direction is a minor direction and for which the splitting takes place in the intervals of ranks $1, 2, \ldots, N$.

B. If the mentioned semineighborhoods u_{pr} and u_{m} are small enough, then the intervals in the construction of A will not intersect except for those which coincide by construction (on l' and l'').

These assertions are consequences of Lemma 13 of §7.

The segment $2s_n$ of the direction selected above, is attached to p_n in the neighborhood P of p_n which is now chosen in such a way that the following three requirements are satisfied:

1) *The oscillation of each function* $g_k^n(x_k)$, *which corresponds to* $f \in F$, *in* P *must be less than* $\varepsilon/4$.

2) *The neighborhood* P *must be so small that under the condition that* $s_n \subset P$ *it is possible to map* σ_n *on* s_n *(see §4), and to satisfy the*

requirements 1_{n+1}, 2_{n+1} (§§4, 6).

3) *The projection of* $2s_n$ *on* l' *and* l'' *along the principal and minor directions must fall within the above constructed (see assertions* A *and* B) *semineighborhoods* u_{pr} *and* u_m *of the point* p_n *on* q_n *if* $2s_n \subset P$.

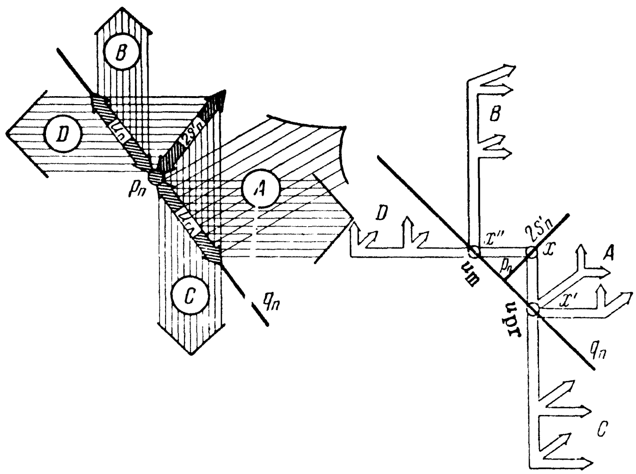

Figure 18. A double generating scheme of the interval $2s_n'$ on $D_{n+1} \cup 2s_n$. On the left, the first layers are shaded; on the right the representation is more schematic.

A sufficiently small neighborhood P of the point p_n will satisfy the requirement 1) because of the equicontinuity of the functions $f \in F$, the continuous dependence of $g_k^n(x_k)$ on $f \in F$, and the possibility of applying the Arzela-Ascoli lemma to the functions $g_k^n(x_k)$ and $f \in F$. Earlier (§§4, 6) it was established that for a sufficiently small P the requirement 2) is satisfied. Finally, the possibility of fulfilling the requirement 3) for small enough neighborhoods P is a consequence of the assertions A and B.

Now we select a neighborhood P that satisfies the requirements 1), 2), and 3). In P we pick $2s_n$ with the above chosen direction. We construct the mapping σ_n on s_n as in §4. On $D_{n+1} = D_n \cup s_n$ the conditions 1_{n+1} and 2_{n+1} are fulfilled because of 2).

Let us now construct on $D_n \cup 2s_n$ (Figure 18) a double generating scheme of the interval $2s_n$ of the following structure:

1. The initial interval $2s_n$ has two generating layers whose directions are the principal and the minor ones.

2. From the interval of the first rank, which lies on the principal direction, there starts a scheme A (see assertion A). From the remaining

intervals of the first rank to which the layer of the first direction leads, there issues the scheme C (see assertion A).

3. From the intervals of the first rank to which the layer of the minor direction leads, there start in the same way the schemes B (from u_m), and D (from the rest).

This construction is actually a generating scheme (double one). Indeed, the schemes A, B, C, D, and D_n do not intersect (except in the general intervals on u_{pr} and u_m). Since (except for the initial intervals) these schemes do not have intervals on u_{pr} and u_m, their layers of rank greater than 1 do not intersect u_{pr} and u_m, and hence not $2s_n$. The layers of the first rank do not intersect $2s_n$ because of the definitions of the principal and minor directions.

We will call the obtained scheme the large scheme.

Each zigzag of the large scheme which leads away from $2s_n$ either passes through at least N intervals distinct from $2s_n$, where the large scheme splits, or else terminates with a free end of lower rank. In any case, *from all the intervals of rank* $1, 2, \ldots, N$ *in the large scheme, which enter into the schemes A and C, and from the intervals of ranks* $2, 3, \ldots, N+1$ *in the large scheme, which enter into the schemes* B *and* D, *there issue two layers*. This follows from the assertions A and B.

Construction of the functions $g_k^{n+1}(x_k)$. We have seen (see (3)), that on D

$$g^n(x) = \sum_{k=1}^{3} g_k^n(x_k).$$

This formula can be considered to be the definition of $g(x)$ in the coordinate parallelepiped, stretched out over D_n in the product of the regions of definition of the functions $g_k^n(x_k)$ $(k = 1, 2, 3)$. On D_{n+1}, there is defined the function $g^{n+1}(x) = f^{n+1}(x) - f^{nr}(x)$. The function $g_k^{n+1}(x_k)$ is to be found so that on D_{n+1} we would have

$$g^{n+1}(x) = \sum_{k=1}^{3} g_k^{n+1}(x_k). \tag{$*$}$$

In this manner, when $x \in D$, and, in particular, at the point p_n,

$$\sum_{k=1}^{3} g_k^{n+1}(x_k) = g^n(x).$$

We determine $g^{n+1}(x)$ on $2s_n$ so that the function

$$g^{n+1}(x) - g^n(x) = \Delta_0(x) \tag{9}$$

on $2s_n$ be even relative to the middle of this interval. It is obvious that the function $\Delta_0(x)$ is defined and continuous on $D_n \cup \overline{2s}_n$ and is different from zero only on $2s_n$.

We shall determine the functions $g_k^{n+1}(x_k)$ so that the equation (*) is fulfilled everywhere on $D_n \cup \overline{2s}_n$. We can do this by distributing the corrections along two directions that correspond to the larger scheme.

For the zeroth approximation to $g_k^{n+1}(x_k)$ we take ${}^0g_k^{n+1}(x_k) = g_k^n(x_k)$. If one substitutes the zeroth approximation in equation (*) for $g_k^{n+1}(x_k)$, the equation will be destroyed only on $2s_n$. We obtain the first approximation from the zeroth one by making corrections on the intervals of the layers of rank 1 of the large scheme. If $x \in 2s_n$, and if, for example, x_1 and x_2 are points (of these intervals of layers) that correspond to x, we obtain

$$\Delta_1^1(x) = \gamma_1 \Delta_0(x),$$

$$\Delta_2^1(x) = \gamma_2 \Delta_0(x).$$

But then if $\gamma_1 + \gamma_2 = 1$, and if

$${}^1g_1^{n+1}(x_1) = {}^0g_1^{n+1}(x_1) + \Delta_1^1(x_1),$$

$${}^1g_2^{n+1}(x_2) = {}^0g_2^{n+1}(x_2) + \Delta_2^1(x_2),$$

$${}^1g_3^{n+1}(x_3) = {}^0g_3^{n+1}(x_3),$$

the equation (*) will be vitiated on the intervals of the first rank only. In general, for the $(i-1)$st approximation the equation (*) will be destroyed on $D_n \cup 2s_n$ only on the intervals of the large scheme of rank $i-1$. The ith approximation is then obtained from the $(i-1)$st one by making corrections on the intervals of layers of rank i of the larger scheme. If x belongs to the layer u of rank $i-1$ of the large scheme, and if, for example, u_2 and u_3 are intervals of layers that issue from u, while $x_2(x) \in u_2$, and $x_3(x) \in u_3$ correspond to x, and if the $(i-1)$st disjoint at the point x is

$$\Delta_{i-1}(x) = g^{n+1}(x) - \sum_{k=1}^{3} {}^{i-1}g_k^{n+1}(x_k), \tag{10}$$

then we set

$$\begin{aligned} \Delta_2^i(x_2(x)) &= \gamma_2 \Delta_{i-1}(x), \\ \Delta_3^i(x_3(x)) &= \gamma_3 \Delta_{i-1}(x), \end{aligned} \tag{11}$$

where $\gamma_2 + \gamma_3 = 1$. (We do not assume that γ_2 and γ_3 are constants. They are functions of x, and will be determined later.) Now we suppose that

$$ {}^{i}g_2^{n+1}(x_2) = {}^{i-1}g_2^{n+1}(x_2) + \Delta_2^i(x_2) \text{ and so on} \tag{12}$$

and that the ith approximation is constructed so that the equation (*) is vitiated only on the intervals of rank i of the large scheme. The process described in §7 is called the distribution of corrections. Thanks to the construction of the large scheme, it proceeds in two directions when $1 \leqslant i \leqslant N$ or $2 \leqslant i \leqslant N+1$, and later terminates as in the case of a simple generating scheme when all intervals of some rank remain free.

We still have to take care of γ_1 and γ_2, for every distribution of the corrections, so that (see (7))

$$ | {}^{i}g_k^{n+1} | < 3 + \theta_{n+1}$$

and all corrections $\Delta_k^l(x_k)$ will be continuous, will vanish at the ends of the intervals of the layers of the large scheme, and will depend continuously on x and $f \in F$. Under these conditions the equation (*), i.e. (6), will be satisfied because of the results of the lemmas of §7; and, in view of (5), (6), (7), and (8), the fulfillment of the conditions 3_{n+1} and 4_{n+1} will have been established.

Lemma 16. *Suppose that the layer of the direction x_1 leads to the interval u of rank $i \geqslant 1$ of the large scheme, and that the layers of the directions x_2 and x_3 lead away from it. Let $x \in u$. Then*

$$ | {}^{i-1}g_k^{n+1}(x_k) | < 3 + \theta_n \quad (k = 1, 2, 3), \tag{13}$$

$$ \left| \sum_{k=1}^{3} {}^{i-1}g_k^{n+1}(x_k) \right| \leqslant 1. \tag{14}$$

Proof. Since u is an interval of rank i, it has not been touched previously in the distribution of the corrections: ${}^{i-1}g_k^n(x_k) = g_k^n(x_k)$. Hence (13) follows from (4), while (14) follows from the estimate of $g^n(x)$ (see definition $g^n(x)$).

Lemma 17. *In the hypotheses of Lemma 16, let $\Delta_{i-1}(x)$ be continuous on $\overline{u}$, vanishing at the ends of the disjoint u (see (10)), and depend continuously on $f \in F$. Furthermore, suppose that*

$$ | \Delta_{i-1}(x) | \leqslant 1 + \varepsilon.$$

Under these conditions one can find corrections $\Delta_2^i(x)$, $\Delta_3^i(x)$ so that

1) $|\Delta_2^i(x)|$, $|\Delta_3^i(x)| < \max\left(\left||\Delta_{i-1}(x)| - \frac{\varepsilon^2}{30}\right|, \varepsilon\right)$,

2) $|{}^i g_2^{n+1}(x_2)|$, $|{}^i g_3^{n+1}(x_3)| < 3 + \theta_{n+1}$,

3) $\Delta_2^i(x) + \Delta_3^i(x) = -\Delta_{i-1}(x)$,

4) $\Delta_2^i(x)$ *and* $\Delta_3^i(x)$ *will depend continuously on* $f \in F$, *and, when* $\Delta_{i-1}(x) \to 0$, $\Delta_2^i(x) \to 0$ *and* $\Delta_3^i(x) \to 0$. *(Here it is assumed in accordance with (12), that* ${}^i g_k^{n+1}(x_k) = {}^{i-1}g_k^{n+1}(x_k) + \Delta_i^k(x_k(x))$.*)*

Proof. The numbers $a = {}^{i-1}g_1^{n+1}(x_1)$, $b = {}^{i-1}g_2^{n+1}(x_2)$, $c = {}^{i-1}g_3^{n+1}(x_3)$, $d = g^n(x)$ (by Lemma 16), and $s = \Delta_{i-1}(x)$, $\theta = \theta_n$, and ε satisfy the conditions of the arithmetic Lemma 14. The conclusions of that lemma coincide in these notations with the conditions of the present lemma if one sets

$$\Delta_2^i(x) = \Delta b, \quad \Delta_3^i(x) = \Delta c.$$

Remark. It is obvious that Lemmas 16 and 17 remain valid if one makes a permutation of x_1, x_2, x_3 in their hypotheses and conclusions.

Lemma 18. *If the first disjoints* $\Delta_0(x)$, $\Delta_1(x)$, $\Delta_2(x)$ *do not exceed* $1+\varepsilon$:

$$|\Delta_0(x)| \leqslant 1+\varepsilon, \quad |\Delta_1(x)| \leqslant 1+\varepsilon, \quad |\Delta_2(x)| \leqslant 1+\varepsilon,$$

and if the functions of the first and second approximations ${}^1g_k^{n+1}(x_k)$, ${}^2g_k^{n+1}(x_k)$ *are less than* $3+\theta_{n+1}$:

$$|{}^1g_k^{n+1}(x_k)| < 3+\theta_{n+1}, \quad |{}^2g_k^{n+1}(x_k)| < 3+\theta_{n+1},$$

then one can find $g_k^{n+1}(x_k)$,

$$|g_k^{n+1}(x_k)| < 3+\theta_{n+1},$$

so that the equation (*) *will be satisfied. If* $\Delta_0(x)$ *and* $\Delta_1(x)$, ${}^1g_k^{n+1}$ *and* ${}^2g_k^{n+1}$, *depend continuously on* x *and* $f \in F$, *then* $g_k^{n+1}(x_k)$ *can be selected to be continuously dependent on* x *and* $f \in F$.

Proof. The Lemma 17 is in this case applicable to all intervals of the large scheme whose rank is greater than zero and from which issue (lead away) two layers. This is true, because in the use of Lemma 17 for the distribution of corrections the Δ_i decreases only when i increases. Making use of the conclusion 1) of Lemma 17, we see that if from the beginning of the large scheme up to a given one of its intervals there have been N intervals from which issued two layers, then in this distribution of corrections the quantity Δ_i is less than $\max(|1+\varepsilon - N\varepsilon^2/30|, \varepsilon)$. But in the large scheme each zigzag with a free end either has at least N first intervals from which two layers issue, not counting the beginning, or all intervals of the zigzag up to the free one, included, have two issuing layers. Bearing in mind that $N\varepsilon^2/30 > 1$, we see that in both cases all corrections Δ_{n+1} are in absolute value less than ε. In the further distribution of the corrections with the aid of simple generating schemes of intervals of rank $N+1$, as in Lemma 8 within §7, the functions $g_k^n(x_k)$ will receive corrections whose absolute value is less than ε, on the new intervals. But on these intervals

$$|g_k^{n+1}(x_k)| < |g_k^n(x_k)| + \varepsilon < 3 + \theta_n + \varepsilon < 3 + \theta_{n+1},$$

and since on the intervals of lower rank the inequality follows from Lemma 17 (rank > 1) and from the hypothesis of Lemma 18 (rank 0 and 1), the latter lemma is proved.

If one now determines Δ_0, Δ_1, Δ_2, ${}^1g_k^{n+1}$, ${}^2g_k^{n+1}$ so that they satisfy the conditions of Lemma 18, then, obviously, the construction of the function g_k^{n+1} under the requirements 3_{n+1} and 4_{n+1} will have been accomplished. Let us first consider the distribution and corrections from the interval of the zeroth rank $2s_n$. Here $\Delta_0(x) = g^{n+1}(x) - g^n(x)$, ${}^0g_k^{n+1}(x_k) = g_k^n(x_k)$, $\Delta_0(x)$ depends continuously on x and f, and vanishes at the ends $2s_n$ of the disjoint. For the sake of definiteness, let us assume that the coordinates of the principal and minor directions of the point $x \in 2s_n$ are x_1 and x_2. Let u_1 and u_2 be the corresponding intervals of the first rank of the large scheme, and let $x' \in u_1$, $x'' \in u_2$ be points which correspond to x (Figure 18). We will write also $x_1(x)$, $x_2(x)$, $x(x_1)$, $x(x_2)$, $x(x')$, $x_2(x_1')$, etc. to indicate this correspondence.

Lemma 19. *If the point x lies in the above-defined neighborhood P of the point p_n, then*

$$\left|\sum_{k=1}^{3} g_k^n(x_k)\right| < 1 + \varepsilon;$$

if $x \in 2s_n$, then

$$|\Delta_0(x)| = |g^{n+1}(x) - g^n(x)| < 1 + \varepsilon.$$

Proof. At the point $p_n = (p_{n_1}, p_{n_2}, p_{n_3}) \in D_n$

$$\sum_{k=1}^{3} g_k^n(p_{n_k}) = g^n(p_n)$$

(see definition $g^n(x)$),

$$|g^n(p_n)| \leqslant 1.$$

Because of the conditions on the neighborhood P, we find that in it

$$| g_k^n(x_k) - g_k^n(p_{n_k}) | < \frac{\varepsilon}{4}.$$

Using this and the preceding inequality, we obtain the first conclusion of the lemma.

The function $| g^{n+1}(x) - g^n(x) |$ is even (see the definition of $g^n(x)$) on $2s_n$, and it vanishes at the endpoints of this segment. Therefore, it will be sufficient to establish the second conclusion of the lemma on s_n.

By the definition of g^n we have

$$g^{n+1}(p_n) - g^n(p_n) = 0$$

and

$$| g^{n+1}(x) - g^{n+1}(p_n) | < 1.$$

The first requirement on P guarantees the fulfillment of the inequality

$$| g^n(x) - g^n(p_n) | < \frac{3}{4}\varepsilon,$$

which together with the preceding inequality proves Lemma 19.

Lemma 20. *For every* $x \in 2s_n$ *one can find* $\Delta_1^1(x)$ *and* $\Delta_2^1(x)$ [*we will write also* $\Delta_1^1(x_1)$ *and* $\Delta_2^1(x_2)$ *for* $\Delta_1^1(x(x_1))$ *and* $\Delta_2^1(x(x_2))$ *respectively*] *such that*

1) $\Delta_1^1(x) + \Delta_2^1(x) = \Delta_0(x)$,
2) $| {}^1g_k^{n+1}(x_k) | = | {}^0g_k^{n+1}(x_k) + \Delta_k^1(x_k) | < 3 + \theta_{n+1}$,
3) $| {}^0g_1^{n+1}(x_1) - \Delta_2^1(x(x_1)) | < 3 + \theta_n + \frac{\varepsilon}{2}$,
4) $\Delta_1^1(x)$ *and* $\Delta_2^1(x)$ *depend continuously on* x *and* $\Delta_0(x)$, *and when* $\Delta_0(x) \to 0$ *so does* $\Delta_k^1(x) \to 0$.

Proof. *The numbers*

$$a = {}^0g_1^{n+1}(x_1), \quad b = {}^0g_2^{n+1}(x_2), \quad c = {}^0g_3^{n+1}(x_3),$$

$$s = g^{n+1}(x), \quad \theta = \theta_n \text{ and } \varepsilon$$

satisfy (because of the fulfillment of condition 4_n and by the definition of θ_n and ε in Lemma 19) all the requirements of the arithmetic Lemma 15. Applying it, we obtain the conclusion of Lemma 20 if we set

$$\Delta_1^1(x) = \Delta a, \quad \Delta_2^1(x) = \Delta b.$$

In particular, for this definition of Δ_k^1 and ${}^1g_k^{n+1}$, we have

$$| \Delta_1(x) | < 1 + \varepsilon \text{ and } | {}^1g_k^{n+1}(x_k) | < 3 + \theta_{n+1}.$$

In order that the condition of Lemma 18 be satisfied, it is still necessary to determine Δ_k^2 and ${}^2g_k^{n+1}$ so that $| \Delta_3(x) | \leqslant 1 + \varepsilon$ and

$|{}^2g_k^{n+1}(x_k)| < 3 + \theta_{n+1}$. For those intervals of the large scheme where it splits, i.e. for all, except $u_2 \subseteq u_m$, this can be done with the aid of Lemma 17.

We introduced the point $x''(x)$ with the coordinates x_k'', whereby the point x'' and its coordinates are functions (linear) of the point x, or of any of its coordinates, and conversely. We thus have

$$ {}^1g_2^{n+1}(x_2'') = {}^0g_2^{n+1}(x_2'') + \Delta_2^1(x_2'') \quad (x_2'' = x_2). $$

The remaining functions of the first approximation coincide with the functions of the zeroth approximation. Let us suppose that in accord with the distribution of the corrections along the directions of the large scheme,

$$ {}^2g_1^{n+1}(x_1'') = {}^0g_1^{n+1}(x_1'') + \Delta_1^2(x_1''), \text{ where } \Delta_1^2(x_1'') = -\Delta_2^1(x(x_1'')). $$

Because of the choice of $\Delta_2^1(x)$ (see Lemma 20),

$$ |\Delta_1^2(x_1'')| \leqslant 1 + \varepsilon. $$

Lemma 21. *In terms of the above notation*

$$ |{}^2g_1^{n+1}(x_1'')| < 3 + \theta_{n+1}. $$

Proof. According to conclusion 3) of Lemma 20,

$$ |{}^0g_1^{n+1}(x_1) - \Delta_2^1(x(x_1))| < 3 + \theta_n + \frac{\varepsilon}{2}, $$

where x_1 is the coordinate of an arbitrary point $x \in 2s_n$, in particular $x(x'')$. In view of the first requirement on P (and u'', obviously, lies in P),

$$ |{}^0g_1^{n+1}(x_1) - {}^0g_1^{n+1}(x_1'')| < \frac{\varepsilon}{4}. $$

Whence,

$$ |{}^0g_1^{n+1}(x_1'') - \Delta_2^1(x(x_1))| < 3 + \theta_n + \frac{3}{4}\varepsilon < 3 + \theta_{n+1}, $$

which was to be proved, because $\Delta_1^2(x_1'') = -\Delta_2^1(x(x_1))$.

Since each successive correction does not exceed, in the above described process, the preceding disjoints, we obtain from the mentioned fact that $|\Delta_1^2(x_1'')| \leqslant 1 + \varepsilon$, the result that $|\Delta_3(x)| \leqslant 1 + \varepsilon$. Bearing in mind Lemma 21, we can convince ourselves that our chosen $\Delta_k^2(x_k)$ does, indeed, satisfy the conditions of Lemma 18. This lemma has been proved, and we obtain functions $g_k^{n+1}(x_k)$ that fulfil all the requirements that were stated in the beginning of this section, and the inequalities (6) and (7). If we suppose (see (8)) that

$$f_k^{n+1}(x_k) = f_k^{n_r}(x_k) + g_k^{n+1}(x_k),$$

then we obtain a decomposition which has the properties 3_{n+1}, 4_{n+1}.

This completes the proof of the inductive lemma, because for $n = 1$, it is trivial.

Thus, the tree $X = \overline{\bigcup_{n=1}^{\infty} D_n}$, the homeomorphism of X on Ξ, and the decomposition of a function from F into the sum of functions of the coordinates on D_n have been constructed under the requirements of the inductive lemma.

§10. Proof of Theorem 3

As a result of the application of the processes described in the preceding section, one obtains trees D_n that are realizations of Δ_n, where $X = \overline{\bigcup_{n=1}^{\infty} D_n}$ realizes Ξ in the form of a subset of the three-dimensional space.

On each tree, every function $f \in F$ can be represented as

$$f(x) = \sum_{k=1}^{3} f_k^n(x_k),$$

where the continuous functions f_k^n of the coordinates x_k of the point $x \in D$ depend continuously on F. The sequence $f_k^n(x_k)$ converges uniformly as $n \to \infty$. This follows from the fact that $|f_k^n(x_k) - f_k^{n_r}(x_k)|$ is not greater than $4/r^2$ when $n_r < n \leqslant n_{r+1}$ and, hence,

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| < \sum_{l=r}^{\infty} \frac{1}{l^2} \quad (n > n_r).$$

Let us denote by $f_k(x_k)$ the limits of these sequences. The sum of these three functions is a continuous function $f(x_1, x_2, x_3)$. For the point $(x_1, x_2, x_3) \in D_n$,

$$\sum_{k=1}^{3} f_k^m(x_k) = f(x) \text{ for all } m \geqslant n.$$

Therefore we have also for the limit the result

$$\sum_{k=1}^{3} f_k(x_k) = f(x) \quad \text{at each point} \quad x \in D_n \quad \text{for any} \quad n.$$

But $\bigcup_{n=1}^{\infty} D_n$ is an everywhere dense subset of its closure in X. The continuous functions $f(x)$ and $\sum_{k=1}^{3} f_k(x_k)$ coincide, therefore, on the entire tree X.

The proof will be complete if we can establish the continuous dependence of $f_k(x_k)$ on f.

Let $\varepsilon > 0$ be given. Let us consider an N so large that $|f_k^n(x_k) - f_k(x_k)| < \varepsilon/3$ for all $n \geqslant N$ and for all f_k^n, f_k which correspond to any function $f \in F$.

In view of the requirement 3_n, the functions $f_k^n(x_k)$, with a fixed $n = N$, depend continuously on $f \in F$. Therefore, f has a neighborhood of radius δ such that for $f' \in F$ and $|f' - f| < \delta$ it is true that $|f_k'^N(x_k) - f_k^N(x_k)| < \varepsilon/3$ for all x_k. From this it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|f' - f| < \delta$, then $|f_k'(x_k) - f_k(x_k)| < \varepsilon$, which was to be proved.

In this manner, *for every family* F *of real equi-continuous functions* $f(\xi)$ *defined on a tree* Ξ, *each of whose points has a branching index less than or equal to* 3, *one can realize the tree in the form of a subset* X *of the three-dimensional cube* E^3 *in such a way that every function of the family* F *can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1, x_2, x_3)$ *is the image of* $\xi \in \Xi$ *in the tree* X, *the* $f_k(x_k)$ *are continuous real functions of a single variable, and* f_k *depends continuously on* f *in the sense of uniform convergence.*

This is Theorem 3.

It implies Theorem 1, as was indicated in the Introduction.

APPENDIX

The space of the components of the level sets of a continuous function

That the set of the components of the level sets of a continuous function, defined on a square, is a tree is clear from Figure 19. Here we will assign an exact meaning to these words by following A.C. Kronrod [4] who introduced the concept of the space of the components of level sets, and K. Menger [3] who has made a study of trees. The theorems proved below are the main tools in both parts of the work. At the end of the Appendix there is placed (for the nonspecialists) a list of the basic concepts of point-set topology.

A. Construction of the metric space T_f

Let a continuous real function $f(a)$ be given on a continuum A (Figure 19). The set of a level, or a level set, is the set of all points a for which $f(a)$ has the same value. The set of a level is thus a closed set; the level sets do not intersect, and constitute all of A. Each set of a given level consists of components, continua that do not intersect each other.

Let us consider the entire set T_f of all components of all level sets of the continuous function $f(a)$. T_f will be called the space of components of the level sets of $f(a)$. We define a metric on this space so that T_f becomes a metric space. The components of the level sets of $f(a)$ are subsets of A and are points in T_f. Any given component will be denoted, the first time, by a capital letter, and after that by the same small letter.

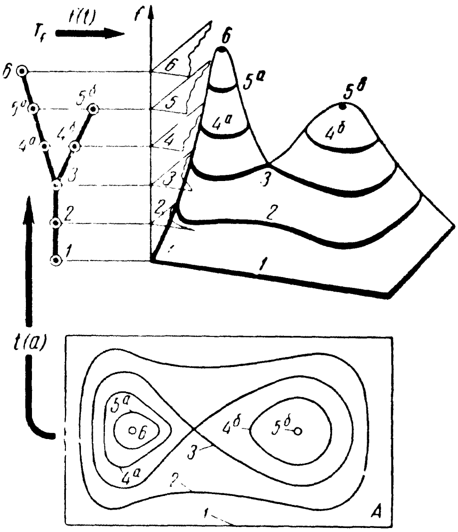

Figure 19. The set of levels, the space of components, and the graph of the function. Some components are denoted by numbers. The branching index of the points 1, 5^{σ}, $6 \in T_f$ is 1, of the points 2, 4^{a}, 4^{σ}, 5^{a} is 2, of the point 3 is 3. The corresponding components thus do not divide A, divide A into 2 parts, or into three parts, respectively.

As is known, the oscillation of a function on a set is the difference between its upper boundary and its lower boundary on the given set. The oscillation of a continuous function on a compact is finite and non-negative.

Let K_1 and K_2 be components of a level set of a continuous function $f(a)$ on a continuum A. By $\mathrm{P}(K_1,K_2)$, we denote the lower boundary of the oscillation $f(a)$ on all continua $F \subseteq A$ that contain K_1 and K_2:

$$\mathrm{P}(K_1, K_2) = \inf_{K_1 \cup K_2 \subset F \subset A} [\max_{a \in F} f(a) - \min_{a \in F} f(a)].$$

If one now defines the distance between points k_1 and k_2 of the space of components as $\rho(k_1,k_2) = \mathrm{P}(K_1,K_2)$, then T_f becomes a metric space. It is, indeed, obvious that

$$0 \leqslant \rho(k_1, k_2) = \rho(k_2, k_1) \leqslant \rho(k_1, k_3) + \rho(k_3, k_2).$$

In order to prove that $\rho(k_1,k_2) = 0$ implies $k_1 = k_2$, we have to make use of the next lemma.

Lemma 1. *For every open set* E $(E \subseteq A)$ *which contains a component* K *of a level set of a function* $f(a)$ *that is continuous on the continuum* A, *there exists a* $\delta > 0$ *such that if* $\rho(k, k_1) < \delta$, *then the component* K_1 *is contained in* E.

Proof. If the lemma were not true (Figure 20), there would exist a sequence of components K_n such that $\rho(k, k_n) < 1/n$ even though, for every n, K_n would contain a point b_n exterior to E. But by the definition of $\rho(k, k_n)$ there exist for $n = 1, 2, \ldots$ continua $F_n \subseteq A$, each of which contains K and K_n with the same n, and such that the oscillation of $f(a)$ on F_n is less than $2/n$. Therefore, the values of f on F_n must differ from the values of $f(a)$ at the points of K by less than $2/n$. The sequence of the points b_n $(n = 1, 2, \ldots)$ that are exterior to E have, because of the compactness of $A \setminus E$, a limit point $b \in A \setminus E$. The lower topological limit $\underline{\mathrm{lt}}\, F_n$ of the connected subsets F_n of the compact A is not empty, since it contains K. Thus the upper topological limit $\overline{\mathrm{lt}}\, F_n$ is connected. At the points of the upper limit, $f(a)$ takes on the same value as on K, because in every neighborhood of such a point there are points of F_n for every n (no matter how large), but these $f(a)$ will differ from $f(a)$ $(a \in K)$ by less than $2/n$.

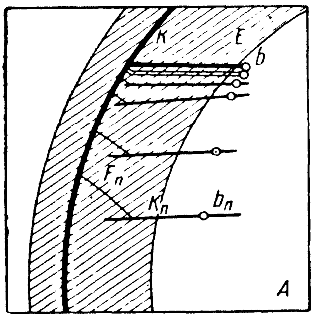

Figure 20. For Lemma 1. If for every $\rho(k, k_n)$ the components K_n have points b_n exterior to E, then K will have a point b exterior to E. The heavy curve is $\overline{\mathrm{lt}}F_n$.

The upper limit, obviously, contains also $K \subset E$ and $b \in A \setminus E$. This contradicts the fact that K is a component contained in E, because the upper limit, a connected set where $f(a)$ is constant, must lie entirely in one component. This establishes the lemma.

On the basis of Lemma 1, it follows from $\rho(k_1, k_2) = 0$ that K_1 and K_2 both lie in any given open set if this set contains either K_1 or K_2. But this can happen only if $K_1 = K_2$ because otherwise the distance between K_1 and K_2 in A would be positive.

The metric in T_f has thus been defined. The topology induced by this metric in T_f coincides with that of the work [4] if A is locally connected. A.S. Kronrod introduces a topology in T_f with the aid of neighborhoods which are defined as sets K that intersect with some open sets $E \subseteq A$. It can be easily seen that the topology on T_f depends only on the decomposition of A into components.

B. Two representations connected with a continuous function

Let us consider two representations, or mappings (Figure 19):

1. $t(a)$ *maps* A *on* T_f *and mates any point* a *of the continuum* A *with the point* $t \in T_f$, *where* t *is the component* $T \subseteq A$ *which contains* a.

2. $f(t)$ *maps* T_f *into the real axis* f *and mates any point* $t \in T_f$ *with a number* f, *the value of* $f(a)$ *at the points of the component* $T \subseteq A$ *that corresponds to* $t \in T_f$.

The use of the same letter f for $f(a)$ and $f(t)$ should not lead to any misunderstanding because these functions have entirely different definitions. We will say that the function $f(a)$ defined on A generates the function $f(t)$ on T_f.

If A *is locally connected, then each of these mappings is continuous.*

1. Since $f(a)$ is continuous, it is true that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the oscillation of $f(a)$ on any set of diameter less than δ is less than ε. Because of the local connectedness of A, any δ-neighborhood of a point $a \in A$ has a connected subneighborhood $u_\delta(a)$. Obviously, if b is contained in $u_\delta(a)$, the components K_a and K_b of the level sets that contain a and b are such that $\rho(k_a, k_b) < \varepsilon$.

2. If k_1, k_2 are two points of T_f that correspond to K_1, K_2, and if $\rho(k_1, k_2) < \varepsilon$, $a_1 \in K_1$, $a_2 \in K_2$, then $|f(a_1) - f(a_2)| < \varepsilon$, because the oscillation of a function is not less than its increment. Thus, $|f(k_1) - f(k_2)| < \varepsilon$.

The continuity of $t(a)$ and $f(t)$ has thus been proved.

If on A there is given a continuous function $g(a)$ which is constant on each component of every level set of the function $f(a)$, then $g(a)$ also generates a *continuous* function $g(t)$ on T_f (namely one which is equal to $g(a)$ at each point of the corresponding component), and we have $g(t(a)) = g(a)$. Indeed, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the oscillation of $g(a)$ on any set of diameter less than δ is less than ε. Let $E_\delta(T)$ be a δ-neighborhood of the component $T \subseteq A$, i.e. the set of points of A all of whose points are nearer than a distance δ from T. By Lemma 1, $t(T)$ has in T_f a neighborhood all of whose components lie in the interior of $E_\delta(T)$. Hence, we have found, for the given $\varepsilon > 0$, a neighborhood of the point $t \in T_f$ in which $|g(t) - g(t_1)| < \varepsilon$. This establishes the continuity of $g(t)$.

Let us now consider the counterimages of points for the mappings $t(a)$ and $f(t)$. The counterimage $t \in T_f$ is a component $T \subseteq A$, i.e., a connected set.

Definition [7]. A continuous mapping is said to be *monotone* if the counterimage of every point is connected.

By means of a monotone mapping one can transform a square with its boundary into a sphere, but not into a torus as we will see later. A monotone transformation is, so to speak, a contraction without "gluing together". Under monotone mappings there are preserved certain topological properties of sets. It is for this reason that the monotoneness of $t(a)$ yields some information on the space T_f.

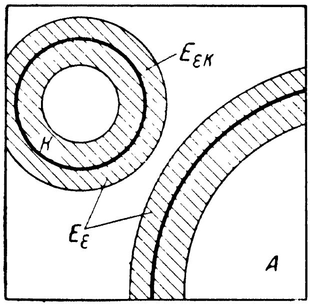

Figure 21. To Lemma 2. The construction of the neighborhood $E_{\varepsilon T}$ of the component T.

In the mapping $f(t)$, the counterimage of a point is the set of all points T_f where $f(t)$ takes on one value, i.e. the set of all components of a set of one level of $f(a)$.

From here on, A will be assumed to be locally connected, so that the functions $t(a)$ and $f(t)$ are continuous.

Lemma 2. *Every point* $t \in T_f$ *has a neighborhood* $u(t)$ *as small as we please (i.e. for every open subset* $E \subset T_f$ *that contains* t, *there exists an open set* $u(t)$, $t \in u(t) \subset E$) *such that its boundary consists of some points of two level sets of* $f(t)$.

Proof. Let T be the component that corresponds to t, and let α be the value of $f(a)$ at the points of T. Let us consider (see Figure 21) the open set E_ε of all points $a \in A$, where $|f(a) - \alpha| < \varepsilon$. E_ε contains T, and let $E_{\varepsilon T}$ denote the component of E_ε that contains T ($E_{\varepsilon T}$ is a region because A is locally connected. If a point lies in $E_{\varepsilon T}$, then the entire component containing this point of the level set $f(a)$ will, obviously, lie in $E_{\varepsilon T}$. It is clear that on the boundary of $E_{\varepsilon T}$, $f(a) = \alpha \pm \varepsilon$. We shall show that the image $u_\varepsilon(t)$ of the region $E_{\varepsilon T}$ under the mapping $t(a)$ satisfies the requirements of Lemma 2 for a small enough positive ε.

1. $u_\varepsilon(t)$ *is an open set in* T_f *that contains* $t \in T_f$.

This assertion is established by the application of Lemma 1 to $E_{\varepsilon T}$ and to the components contained in this region.

2. Suppose that K is a component which under the mapping $t(a)$ is transformed into one of the boundary points of $u_\varepsilon(t)$; then K *is contained in the boundary of* $E_{\varepsilon T}$.

The truth of this assertion can be proved by the application of Lemma 1 to the regions containing K.

3. For a sufficiently small positive ε, the oscillation of the function

$f(a)$ on E_ε, and on the continuum $\overline{E_{\varepsilon T}}$ is as small as we please. This implies that for a positive ε, small enough, $u_\varepsilon(t)$ *is an arbitrarily small neighborhood of* t.

This proves Lemma 2.

It follows from Lemma 2 that a level set of the function $f(t)$ is a zero-dimensional subset of T_f, since each of its points has an arbitrarily small neighborhood whose boundary is not intersected by the level set.

We have thus proved the next theorem.

Theorem 1. *The real continuous function* $f(a)$, *defined on a locally connected continuum* A *is the product of two continuous mappings: a monotone mapping* $t(a)$ *of the continuum* A *on the space* T_f *of the components of the level sets of the functions* $f(a)$, *and a mapping* $f(t)$ *of the space* T_f *on the real axis, under which the counterimage of every point* f *is of zero dimension. The function* $g(a)$, *which is continuous on* A *and constant on each component of the set of the level* $f(a)$, *generates a function* $g(t)$ *continuous on* T_f *such that* $g(a) = g(t(a))$.

C. Singly connected sets

Definition. A locally connected continuum M is said to be *singly connected* [7] if it cannot be represented as the sum of two continua whose intersection is not connected.

For example, the circle and the torus are not singly connected.

Remark. This definition is equivalent to the following ones.

A locally connected continuum is singly connected if every compact subset of it that divides it has a component that divides it.

A locally connected continuum is singly connected if every continuous mapping of it on a circle is homotopic to a mapping on a point.

It does not follow from singly connectedness that every simple closed curve on M can be contracted, without breaking it, into a single point.

Lemma 3 [7]. *The monotone image* F_2 *of a locally connected continuum* F_1 *is a singly connected, locally connected continuum.*

Lemma 4 [7]. *Under a monotone mapping of a compact, the complete counterimage of a continuum is a continuum.*

Proof of Lemma 4. In the opposite case, this complete counterimage could be divided into two nonintersecting closed sets A and B, whose images A' and B' would intersect. If C' were a point of intersection of the images, then its counterimage would intersect A and B, while at the same time it would lie in $A \cup B$, and hence would not be connected. Therefore, the mapping would not be monotone.

Proof of Lemma 3. F_2, the continuous image of a locally connected continuum, is a locally connected continuum. Let A_2 and B_2 be continua in F_2, $A_2 \cup B_2 = F_2$. In view of Lemma 4, the counterimages of A_2 and B_2, the sets A_1 and B_1, are continua. Obviously, $A_1 \cup B_1 = F_1$. Therefore, $A_1 \cap B_1$ is connected in view of the singly connectedness of F_1. But $A_2 \cap B_2$ is the image of $A_1 \cap B_1$ and hence is a connected set. This completes the proof of Lemma 3.

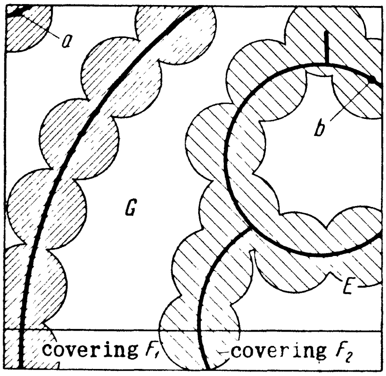

Figure 22. To Lemma 5. If it were true that $A \cup B = E$, $A \cap B = F_1 \cup F_2$, then the region G, which separates the point $a \in F_1$ from the point $b \in F_2$, would intersect the sets A, B that connect a and b. This would contradict that G is connected because $A \cap B = F_1 \cup F_2$ lies in the exterior of G.

Lemma 5 [7]. *The Euclidean cubes of any dimension, and the spheres of dimensions* 2 *and higher, are singly connected.*

Proof. Let us assume the opposite, and suppose, for the sake of definiteness, that the square $E = A \cup B$, where A and B are continua whose intersection $A \cap B$ consists of two nonintersecting compacts, i.e. $A \cap B = F_1 \cup F_2$. Let the distance between F_1 and F_2 be greater than $h > 0$. We will consider spherical neighborhoods with radius $h/3$ of all points of F_1 and F_2. These neighborhoods cover $F_1 \cup F_2$. It is possible to select from them a finite number of neighborhoods, and it is clear that they can be so chosen that F_1 and F_2 are covered, but their coverings do not intersect (Figure 22). It is obvious that the square is broken up by a finite number of curves each of which consists of a finite number of circular arcs, into parts of three types: those which are part of the covering of F_1, those which belong to the covering of F_2 and remaining ones. The coverings of F_1 and F_2 are at a distance greater than $h/3$ from each other. Therefore the remaining regions separate them. Let $a \in F_1$ and $b \in F_2$. Every broken line* that intersects a and b must intersect one of the regions of the remaining points. We consider it an obvious fact for E (a cube or sphere) that among the considered regions there is one G which separates a and b. We note only that this assertion is not true for a torus and other nonsingly connected sets. The continua A and B both contain a and b. Hence G contains

* And, hence, every continuum.

some points of A (which are not in B) and points of B (which are not in A, because $A \cap B = F_1 \cup F_2$). Both sets $A \cap \overline{G}$, $B \cap \overline{G}$ are closed and do not intersect, and their sum is $\overline{G}$, because $A \cup B = E$. This contradicts the connectedness of $\overline{G}$. This contradiction shows that the hypothesis on the incorrectness of Lemma 5 was false. Hence Lemma 5 is true.

By combining Theorem 1 and Lemmas 3 and 5, we obtain the following important property of T_f.

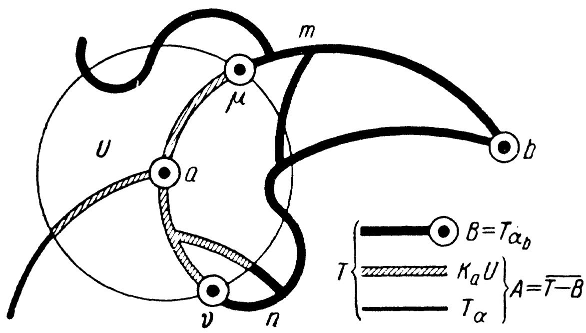

Figure 23. To Lemma 6. The locally connected one-dimensional continuum T that contains the cycle *ambna* can be broken into two connected parts (B is the heavy curve, $A = \overline{T \setminus B}$) by means of a nonconnected intersection.

Theorem 2. *The space of the components of the level sets of a continuous function defined on a singly connected locally connected continuum is a singly connected locally connected continuum. In particular, the space of the components of the level sets of a function that is continuous on a cube of any dimension and on a sphere of dimension greater than* 1 *is such a continuum.*

D. Trees

Definition. A *tree* is a locally connected continuum that does not contain homeomorphic images of a circle [3].

Since a tree is a locally connected continuum, any two points of it can be connected by a closed arc, and since the tree does not contain a homeomorphism of a circle, the arc is unique.

Lemma 6 [7]. *A one-dimensional singly connected continuum is a tree.*

Proof. Let us assume that such a continuum has two points a and b [Figure 23], which can be connected by nonintersecting arcs amb and anb. In view of the one-dimensionality of T, the point a has a neighborhood U, whose closure does not contain b, and whose boundary is of zero dimension.

Let K_aU be the component of the point a in this neighborhood. Because of the local connectedness of T, K_aU is an open set in T. Let us consider $T\setminus K_aU$. This closed set consists of the components, continua T_α, so that $T = (K_aU) \bigcup (\bigcup_\alpha T_\alpha)$. In particular, among these continua there is a component $T_{\alpha_b} \ni b$. Let us suppose that $B = T_{\alpha_b}$ and $A = \overline{T\setminus B}$. Obviously, $A \bigcup B = T$, B is a continuum, and A is a compact. We will show that A is connected. Indeed, from the fact that $T = (K_aU) \bigcup (\bigcup_\alpha T_\alpha)$, it follows that $T\setminus B = (K_aU) \bigcup (\bigcup_{\alpha\neq\alpha_b} T_\alpha) = \bigcup_{\alpha\neq\alpha_b} ((K_aU) \bigcup T_\alpha)$. It is easy to see that each set $(K_aU) \bigcup T_\alpha$ is connected. This implies that $T\setminus B$, and hence A, is connected.

Let us show also that $A \bigcap B$ contains the boundary of U. Indeed $A \bigcap B = \overline{B} \bigcap \overline{T\setminus B}$, i.e. $A \bigcap B$ is the boundary of $B = T_{\alpha_b}$ and, hence, is contained in the boundary K_aU, which is contained in the boundary of U. Each of the arcs amb and anb intersects the boundary of U, since a is in the interior of U, and b is in its exterior. Suppose that μ and ν are the first points of intersection of these arcs with the boundary of U starting from a. $A \bigcap B$ contains μ and ν, since it is obvious that these points are not contained in K_aU, but do lie in B, namely in the boundary of B. From the zero-dimensionality of the boundary of U it follows that $A \bigcap B$ is not connected, because a zero-dimensional connected set cannot have two distinct points. Thus, we have obtained a decomposition of T into the sum of two continua A and B whose intersection is not connected. This means that T is not a singly connected, locally connected continuum. This contradiction to the hypothesis of the lemma proves that T cannot contain homeomorphisms of a circle. Hence, T is a tree, which was to be proved.

Lemma 7. *The space of the components of the level sets of a real continuous function defined on a compact is at most one-dimensional.*

Proof. From Lemma 2 it follows that each point $t \in T_f$ has an arbitrarily small neighborhood whose boundary is contained in the sum of two level sets of $f(t)$ and is, therefore, either empty or zero-dimensional. Therefore, the space T_f is at most one-dimensional.

It is obvious that the space T_f can be zero-dimensional only in the case that the function f is a constant. Eliminating this case, when T_f is a single point, we can draw the following conclusion from Theorem 2, and from the Lemmas 6 and 7.

Theorem 3 [4]. *The space of the components of the level sets of a real continuous function defined on a locally connected, singly connected continuum. is a tree.*

The space of the components of the level sets of a real continuous function defined on an n-dimensional cube or on a sphere of dimension $n \geqslant 2$ *is a tree.*

The *branching index of a point of a tree* is the number* of parts (components) into which the tree falls after the given point is removed from the tree.

If the tree T is the space of the components of the level sets of a continuous function, then the branching index of a point of the tree is related to the structure of the component to which this point belongs.

Theorem 3[a] [4]. *The number of parts into which a component of a level set of a continuous function divides the region of definition of this function is equal to the branching index of the corresponding point of the space of the components.*

Proof. Indeed, the mapping $f(a)$ sets up a single-valued correspondence between the region of definition of the function f and the space of the components (Figure 19).

E. Structure of trees

We have seen that any two points of a tree can be connected by means of a simple arc, and by just one exactly. With the aid of this property one can obtain, following Menger [3], a convenient representation of trees, and can study their structure by reducing the investigation to finite trees, i.e. to trees with a finite number of branching points. We will confine ourselves to the consideration of trees which do not have any points with a branching index greater than three, since we use only this type of tree in Parts I and II of the present work.

Let Ξ be a tree whose points have branching indices not greater than 3. From the compact Ξ we pick a denumerable everywhere dense set $A : a_1, a_2, \ldots$ The pair of points a_1, a_2 determines in Ξ a unique simple arc a_1, a_2, which we denote by σ_0. From the remaining points $a_3, a_4, \ldots$ we pick the first point that is not contained in σ_0, and we denote it by $\overline{a}_3$. There is a unique simple arc $a_1\overline{a}_3$ in Ξ. We denote by ρ_1 the point nearest to $\overline{a}_3$ on the arc σ_0. (This point may happen to be a_1 or a_2.) Next, we denote the arc $\overline{a}_3\rho_1$ by σ_1, and, setting $\sigma_0 = \Delta_1$, $\Delta_1 \cup \sigma_1 = \Delta_2$, we see that when $i = 1$, the simple arc σ_i, the point ρ_1 and the finite trees Δ_i, Δ_{i+1} have the following properties:

1_i) $\Delta_{i+1} = \Delta_i \cup \sigma_i$,

2_i) $\sigma_i \cap \Delta_i = \rho_i$,

* Or the power, or cardinal number of the set of parts, if this set is infinite.

3_i) Δ_i contains all points a_k $(k \leqslant i+1)$.

If the finite trees Δ_i $(i = 1, \ldots, n)$ are constructed, and all Δ_{i+1}, Δ_i, σ_i, ρ_i $(i = 1, \ldots, n-1)$ satisfy the conditions 1_i), 2_i), 3_i), then it is easy to construct Δ_{n+1}. For this purpose we select, from the points of A that have not been included in Δ_n, the point with the smallest subscript. Let it be $\bar{a}_{n+2}$. In view of 3_{n+1}) the subscript of this point is greater than n. Hence, if we include it in Δ_{n+1} we guarantee the fulfillment of condition 3_n). The simple arc $a_1\bar{a}_{n+2} \subset \Xi$ that connects these points is uniquely determined. Suppose that ρ_n is the first point from $\bar{a}_{n+2}$ on $a_1\bar{a}_{n+2}$. We denote the simple arc $a_{n+2}\rho_n$ by σ_n. Then the conditions 1_n) and 2_n) are satisfied. In this manner we can determine Δ_n, σ_n, ρ_n for all $n \geqslant 1$, and the conditions 1_n), 2_n), 3_n) are all satisfied.

Each finite tree Δ_n has no point whose branching index is greater than 3. Indeed, in the opposite case there would be four simple arcs ad_r $(r = 1, \ldots, 4)$ that would intersect at a. Let us denote by B_r the set of those points of the tree that can be connected with a by means of simple arcs that intersect the arc ad_r (excluding, obviously, the point a). Such sets, for different r, will intersect each other, because the simple arc that connects two points of Δ_n is unique. The components of the set $\Xi \setminus a$ (which is open in the locally connected continuum Ξ) are open. Hence, any two points of such a component can be connected by a simple arc. This shows that every set B_r constitutes an entire component of $\Xi \setminus a$. Therefore, there should be at least four such components. But this is impossible, because the branching index of every point of the tree is less than 4.

Because of condition 3_i), and of the fact that A is everywhere dense

$$\overline{\bigcup_{n=1}^{\infty} \Delta_n} = \Xi.$$

The subsets $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide Ξ, because $\bigcup_{n=1}^{\infty} \Delta_n$ is connected, and through the addition of some limit points to a connected set, its connectedness is not destroyed. In particular, the points of the set $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide the tree Ξ into separate parts. The points of a tree which do not divide the tree are called ends of the tree.

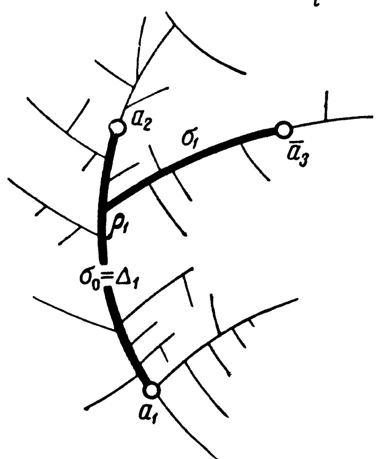

Figure 24. The heavy curve is Δ_2; Δ_1 and Δ_2 do not satisfy the requirement 4) of Lemma 8.

Before we give the conclusions of the study of the structure of a tree, we will change the construction of Δ_n so that the points ρ_n will not be ends of Δ_n. Suppose, for example, that σ_0 has for one of its ends a_2 the point ρ_{n_1}. We join σ_{n_1} to σ_0, and obtain a simple arc which we denote by σ_0^1. If one of the ends of σ_0^1 is ρ_{n_2}, then we join σ_{n_2} to σ_0^1, and obtain the simple arc $\sigma_0^2 = \sigma_0^1 \cup \sigma_{n_2}$, and so on, until either the end σ_0^N is not a ρ_m point for any m, or ad infinitum. In the first case we set $\sigma_0^N = \sigma_0^{\text{new}}$. In the second case, let l be a limit point of the ends σ_0^N. It will not divide Ξ, because it it did, then l would separate a_1 from some point $a_n \in A$,* and l would then belong to one of the sets Δ_n. By the construction of σ_n, l could not be a limit point of ends of Δ_n. It follows that $l \neq \rho_m$ for any m, and we have obtained for this second case that $\sigma_0^{\text{new}} = a_1 l$. After such a treatment of both ends of σ_0, we pick from the arcs σ_n the first one which is not contained entirely in σ_0^{new}, and repeat the same treatment of its ends. Hereby we will not touch the completed arcs; and, continuing this process, we will obtain a new system Δ_n^{new}, ρ_n^{new}, σ_n^{new}, whose elements we will denote simply by Δ_n, ρ_n, σ_n. This system will have, in addition to the properties 1), 2), 3), also the property

4) $\rho_m \neq \rho_n$ if $m \neq n$.**

We have thus proved the following lemma.

Lemma 8. *Every tree Ξ whose points have no branching index greater than 3 can be represented in the form*

$$\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n},$$

where the Δ_n are finite trees composed of arcs σ_n attached at the points ρ_n so that:

1) $\Delta_1 = \sigma_0$,

2) $\Delta_{n+1} = \Delta_n \cup \sigma_n$,

3) $\sigma_n \cap \Delta_n = \rho_n$,

4) $\rho_m \neq \rho_n$ *if* $m \neq n$, *and the points* ρ_n *are not ends of* Δ_n.

One can show that only the points ρ_n have a branching index greater than two, and that Lemma 8 without the condition 4) is true for every tree. This implies the next theorem.

Theorem 4 [3]. *Every tree Ξ consists of a set that is everywhere dense*

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* Because the components of $\Xi \setminus l$ are regions.

** The old ρ's could coincide (Figure 24) if one connected successively two branches to ρ, the end of Δ. The new construction prevents this, and since Δ has no points with branching index greater than 3, property 4) is satisfied.

in Ξ and is composed of the points of an at most denumerable set of simple arcs which do not intersect pair-wise in more than one point, and of a set consisting of the ends of Ξ (which can be everywhere dense in Ξ and have the power of the continuum). The branching index of the points of Ξ is at most denumerable, and greater than two only in a denumerable set of points (namely, at the points of intersection of simple arcs indicated above).

Figure 25. To Theorems 4, 6, 7.

It is obvious that the representation of the tree in the form of Lemma 8 is not unique. The proof of Theorem 4 will not be given here, because this theorem is not being used in the present work.

Let us also consider the structure of the components of the remainder $\Xi \setminus \Delta_N$. This set is open in Ξ; its components are regions, and in each of them any two points can be connected by means of a simple arc, without passing outside the component.

Lemma 9. *Let Ξ, σ_n, ρ_n, Δ_n $(n = 1,2,\dots)$ be the objects defined in Lemma 8. Then the following statements are true.*

1. *The boundary of every component K of the set $\Xi \setminus \Delta_N$ consists of one point, namely of the point ρ_m $(m = m(N,K) \geqslant N)$.*

2. *Any two points of $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ lie in different components of $\Xi \setminus \Delta_N$ for N sufficiently large.*

Proof. 1. Let us suppose that this boundary has two distinct points a, $b \in \Delta_n \cap \overline{K}$ (Figure 26, 1). The points a and b have nonintersecting connected neighborhoods because Ξ is locally connected. Suppose that $a' \in u_a \cap K$ is a point of the first of these neighborhoods u_a, and $b' \in u_b \cap K$ one of the second neighborhood. The points a' and b' can be connected by means of a simple arc which lies entirely in K, while the points a and b belong to Δ_N as points of the boundary of K and can, therefore, be connected by a simple arc ab in Δ_N. The arcs ab and $a'b'$ do not intersect. From the fact that it is possible to connect a and a' by a simple arc in u_a, and b and b' by a simple arc in u_b, we conclude that in Ξ there is a curve $aa'b'ba$ that contains a homeomorph of the circle. Thus, the boundary of K must be a single point.

Since $\bigcup_{n=0}^{\infty} \sigma_n$ is everywhere dense in Ξ (by Lemma 8), there exists an arc σ_n that intersects the region K. Among such arcs, let σ_m be the one with least subscript. Obviously, $m > N$. Since Δ_{m+1} contains this arc (condition 2), Lemma 8), and since Δ_m does not intersect K, σ_m intersects the boundary of K. But this boundary is a single point that belongs to Δ_N and, hence, to Δ_m. Therefore (condition 3), Lemma 8) the truth of the first statement has been established.

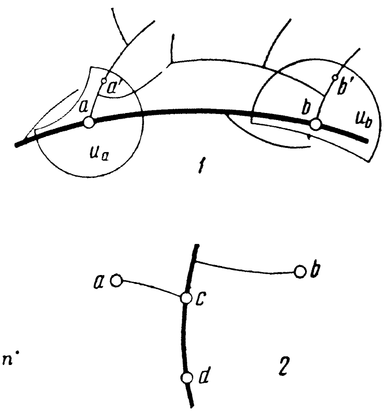


Figure 26. To Lemma 9. The heavy line is the tree Δ_N.

1. If the boundary of a component of the complement of Δ_N had two distinct points a and b, then Ξ would contain a homeomorph of a circle.
2. For sufficiently large N, Δ_N will separate any two points $a, b \in \Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$.

2. Suppose that a and b are two points of $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$, acd and bcd are simple arcs connecting a and b with the point $d \in \Delta_1$, c is the last point away from d that lies on both these arcs (Figure 26, 2). This point can coincide with only one of the points a, b, d, and we can, therefore, assume that $a \neq c$. In this case c separates a from d, for if a and d should belong to the same component of the open set $\Xi \setminus c$, one would be able to connect them by a simple arc not passing through c, and Ξ would contain a homeomorph of the circle, because this arc would not coincide with the simple arc acb. Therefore, $c \in \Delta_N$ for some N because it can be seen from Lemma 8 that the points $\Xi \setminus \bigcup_{n=1}^{\infty} \Delta_n$ do not divide Ξ. This Δ_N separates a from b, for the points a and b can be connected by a simple arc acb, and hence by no other one. This establishes Lemma 9.

F. Realization of trees

All trees can be imbedded homeomorphically in a plane. We construct a planar set that is homeomorphic to a given tree Ξ whose points have branching indices not greater than three. In this we follow Menger [3].

Let $\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}$ be the representation given in Lemma 8. We will select in the plane a straight line segment and an open triangle T_0 containing s_0. Let us map σ_0 on s_0 homeomorphically with the aid of the homeomorphism f_1. Then there will be on s_0 a point p_1 which is the image of ρ_1. We can construct an open triangle T_1, of diameter less than d_1 (this positive number will be defined later) with vertex at p_1, which does not intersect $D_1 = s_0$, except at the point p_1, and whose closure lies in T_0.

We select within T_1 a point and connect it with p_1. Then we obtain a segment s_1. We map σ_1 homeomorphically on s_1. We have constructed a homeomorphism f_2 of Δ_2 on $D_2 = s_0 \bigcup s_1$.

Suppose that we have constructed on R^2 complexes of segments (segment-like complexes) D_i from the segments s_i with the aid of the triangles T_i and the points p_i, and also let f_{i+1} be the homeomorphism Δ_{i+1}, on D_{i+1}, where $i, j = 1, 2, \ldots, n-1$ (see Figure 9) and

1_i) $D_1 = s_0$,

2_i) $D_{i+1} = D_i \bigcup s_i$,

3_i) $D_i \bigcap T_i = p_i$,

4_i) $(R^2 \setminus T_i) \bigcap s_i = p_i$,

5_i) if $i > j$, $\overline{T}_i \bigcap \overline{T}_j = 0$ or else $\overline{T}_i \subset T_j$,

6_i) the diameter T_i is less than $d_i > 0$,

7_i) f_i maps Δ_{i-1} the same way as f_{i-1} $(i > 1)$.

Let the arbitrary positive number d_n be given. On Λ_n there exists, in general, a point $p_n \in \sigma_k$ $(k \leqslant n)$ (if there is no such point, then Λ_n is the resulting tree). The homeomorphism f_n determines, on D_n, a point $p_n \in s_k$, the image of ρ_n. It is easy to select in the triangle T_k a small, open triangle T_n so that the following conditions hold:

1) one of this triangle's vertices is p_n,

2) $\overline{T}_n \subset T_k$,

3) T_n does not intersect s_k,

4) $\overline{T}_n$ does not intersect $\overline{T}_i$ $(i < n)$ if T_k does not lie in T_i,

5) the diameter T_n is smaller than d_n.

Having picked in T_n a point, and connected it to p_n, we obtain a segment which we denote by s_n. Obviously, by mapping σ_n homeomorphically on s_n, we determine the required homeomorphism f_{n+1} on Δ_{n+1} so that the conditions 1_i) to 7_i) will be satisfied. We have thus proved the truth of the following lemma.

Lemma 10. *Let* $\Xi = \overline{\bigcup_{n=1}^{\infty} \Delta_n}$ *be the representation given in Lemma 8. Let* d_n *be a positive number. In the plane* R^2 *one can construct (with the aid of the segments* s_n, *the points* p_n, *and the triangles* T_n) *complexes* D_n *and homeomorphisms* $f_n : \Delta_n \to D_n$ *such that the conditions* $1_n)$-$7_n)$ *are satisfied for any* $n = 2, 3, \ldots$.

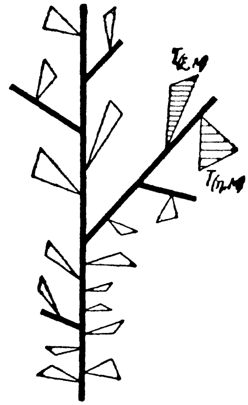
Now, let Ξ, Δ_n, D_n, σ_n, s_n, ρ_n, p_n, T_n, f_n $(n = 1, 2, \ldots)$ be such a system of objects, and suppose that $d_n > 0$, $d_n \to 0$ as $n \to \infty$.

Lemma 11. *In the notation given above,* $X = \bigcup_{n=1}^{\infty} D_n$ *is a tree that is homeomorphic to* Ξ, *and the homeomorphism can be constructed so that it coincides with* f_n *on* Δ_n *if* $n = 1, 2, \ldots$.

Proof. We define a sequence of mappings f_n' $(n = 1, 2, \ldots)$ of Ξ in X, namely on D_n, so that on Δ_n, f_n' coincides with f_n. We obtain f_n' on Ξ as $f_n(\varphi_n(\xi))$: the product of a continuous mapping φ_n of all of Ξ on Δ_n, and f_n which transfers Δ_n on D_n homeomorphically. Such a mapping will coincide with f_n on Δ_n if φ_n keeps every point of Δ_n unchanged. We have, therefore, defined a mapping φ_n on Δ_n so that $\varphi_n(\xi) = \xi$ $(\xi \in \Delta_n)$. Every component $K \subset \Xi \setminus \Delta_n$ has a unique boundary point ρ_m $(m = m(K, n) \geqslant n)$ in accordance with assertion 1 of Lemma 9. Let us set $\varphi_n(\xi) = \rho_{m(K,n)}$ $(\xi \in K)$. Now, $\varphi_n(\xi)$ is everywhere defined; we will show that this mapping is continuous. The point $\xi \in \Xi \setminus \Delta_n$ has a neighborhood K which is transformed into the same point as ξ. We still have to prove the continuity at the points of Δ_n. We will point out a neighborhood for such a point ξ, which will be transformed into an arbitrarily previously given neighborhooa u_ξ. A connected neighborhood $v \subseteq u_\xi$ of the point ξ will do. (This neighborhood exists because of the local connectedness of Ξ.) The points η of this neighborhood of ξ will go into its interior by the transformation φ_n. Indeed, this is obvious for the points $\eta \in \Delta_n$. Let $\eta \in \Xi \setminus \Delta_n$. Then η will be contained in some component K of the set $\Xi \setminus \Delta_n$. Let $\rho = \rho(K, n)$, be the boundary of K. Firstly, $\rho \in v$, because the points η and ξ of the region v can be connected by a simple arc lying in v. On this arc one can find a point of the boundary K because the initial point η of this arc belongs to K while the end ξ does not belong to K; this is the point ρ (Lemma 9). Secondly, the image of ρ under the mapping φ_n is ρ by the definition of φ_n. The continuity of φ_n has thus been proved, and it implies the continuity of $f_n'(\xi) = f_n(\varphi_n(\xi))$.

The sequence of mappings f_n' $(n = 1, 2, \ldots)$ converges uniformly on Ξ. Let a positive ε be given. From the fact that $d_n \to 0$, it follows that for $n \geqslant N(\varepsilon)$ $d_n < \varepsilon$. We will show that at every point $\xi \in \Xi$, and for any $n > N(\varepsilon)$, $\rho(f'_n(\xi), f'_{N(\varepsilon)}(\xi)) < \varepsilon$. This follows from the fact that the image of ξ under the mapping f'_n lies in the triangle $\overline{T}_m$ $(m > N(\varepsilon))$, when $p_n \in D_{N(\varepsilon)}$, or on D_n in accordance with the conditions 1) to 7) of Lemma 10.

Figure 27. To Lemma 11. The heavy line tree D_6 is homeomorphic to Δ_6 that separates ξ and η. Some of the triangles T_m $(m \geqslant 6)$ have been drawn, for which $p_m \in D_6$. Among them $T_{(\xi,M)}$ and $T_{(\eta,M)}$ $(M > 6)$ have been shaded. They contain the images of ξ and η under all mappings f'_m $(m > M)$.



Thus, $f = \lim\limits_{n\to\infty} f'_n$ is a continuous mapping. Obviously, it coincides with f_n on Δ_n. We shall prove that to distinct points of Ξ there correspond distinct images in X. This is obvious for the points $\xi \in \bigcup\limits_{n=1}^{\infty} \Delta_n$. The points ξ and η of $\Xi \setminus \bigcup\limits_{n=1}^{\infty} \Delta_n$ lie, for sufficiently large N, in different components, K_1, K_2 of the complement of Δ_N (Lemma 9). From this, and from the definition of f'_n with the aid of properties 3) and 4) of Lemma 8, it follows that from some M on $(M > N)$ the images ξ, η under f'_m $(m \geqslant M)$ lie in different triangles $T_{(\xi,M)}$, $T_{(\eta,M)}$, whose closures intersect D_N (Figure 27). From the condition 5) of Lemma 10 we now see that $\overline{T}_{(\xi,M)} \cap \overline{T}_{(\eta,M)} = 0$, which shows that $f(\xi) \neq f(\eta)$. In exactly the same way, one can consider the case when $\xi \in \bigcup\limits_{n=1}^{\infty} \Delta_n$, $\eta \in \Xi \setminus \bigcup\limits_{n=1}^{\infty} \Delta_n$. The image of the entire tree Ξ under the mapping f contains all of D_n, and hence it is X. Therefore, f is a reciprocal one-to-one continuous mapping of the compact Ξ on X, i.e. it is a homeomorphism. This implies that X is a tree. Lemma 11 has thus been established.

The process used in the proofs of Lemmas 10 and 11 for the construction of the tree X, and of the mapping f in accord with the conditions 1) to 4) of Lemma 8, for Ξ, Δ_n, σ_n, ρ_n $(n = 1, 2, \ldots)$ and $d_n \to 0$, can be called the *method of attaching branches*. Our result can then be formulated as follows.

Theorem 5 [3]. *Let there be given a tree* Ξ *whose points have no branching indices greater than three; then one can construct in the plane, by the method of attaching branches, a tree* X, *homeomorphic to* Ξ, *and a homeomorphism* f *between* Ξ *and* X.

The next more general theorem can be proved in an analogous way.

Theorem 6 [3]. *Every tree* Ξ *has a homeomorphic image in the plane.*

A set M is said to be universal for a class A_α if each set A_α has a homeomorphic image in M.

Theorem 7 [3]. *If in the representation of Theorem 4 the set of points of intersection of the simple arcs is everywhere dense, and if the branching index of* Ξ *at every one of its points is* n *(respectively, denumerably infinite), then the tree is universal for the class of all trees whose branching index does not exceed* n *(respectively, for all trees). The trees which are described above do actually exist.*

Theorems 6 and 7 are not used in this work. The reader can provide the proofs himself, or he can find them in the work [3]. We note without proof that the space of the components of the level sets of a continuous function defined on a square can be a universal tree. An example (for the case $n = 3$) is the function $F(x, y)$ constructed in Part I (§ 2) of this work.

Concepts and theorems of point-set topology used without further comment

1. **Concepts** ([6], Chapters VII and VIII; [7]; [8]; [9]).

Metric space. Topological space. Open and closed sets, boundary. Continuous mapping and homeomorphism. Everywhere dense set. Connectedness.

A compact is a metric space in which one can select from every infinite sequence a convergent subsequence. A continuum is a connected compact. The component of a point of a set (or simply a component of a set) is the largest connected subset that contains the given point.

A set is locally connected if every neighborhood* of any point contains a subneighborhood of this point.

A set is zero-dimensional if in any neighborhood of each of its points there is a neighborhood of the same point whose boundary is empty.

A set is one-dimensional if in any neighborhood of each of its points there lies a subneighborhood of the same point whose boundary is zero-dimensional.

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* Here and in the sequel, a neighborhood of a point is any open set containing this point.

A region is an open connected set. A simple arc is a set that is homeomorphic to a segment of a straight line. The set A separates B from C if every continuum that contains B and C contains A. If A separates $b \in B \subset M$ from $c \in C \subset M$, then one says that A divides M.

The point x belongs to the upper topological limit $\overline{\mathrm{lt}}\, M_i$ of the sets M_i $(i = 1, 2, \ldots)$ if in every one of its neighborhoods there lie points of an infinite number of the sets M_i. The point belongs to the lower topological limit $\underline{\mathrm{lt}}\, M_i$ if in every one of its neighborhoods there are points of all but a finite number of the sets M_i.

2. **Theorems.**

A metric space which is a continuous image of a compact is a compact, of a continuum is a continuum, of a locally connected continuum is a locally connected continuum [6].

A reciprocal one-to-one continuous mapping of a compact is a homeomorphism [6]. A continuous mapping of a compact is uniformly continuous.

The components of a compact are continua; the components of an open set in a connected space are regions [6].

In a region of a locally connected continuum any two points can be connected by means of a closed arc ([3]; [7]; [9]).

The intersection of a decreasing sequence of continua $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ is a continuum [6].

If the sets $B \subset M$ and $C \subset M$ lie in different components of $M \setminus A$, then A separates B from C. If the closed set A of a locally connected continuum M separates B from C, then B and C lie in different components of $M \setminus A$.

A set that consists of two noncoinciding simple arcs with common ends contains a simple closed arc (homeomorph of a circle). The sum of four simple arcs aa', $a'b'$, $b'b$, ba have the same property if $a'b' \cap ba = 0$ and $aa' \cap bb' = 0$.

In a compact, the upper topological limit of a sequence of connected sets is connected, provided the lower topological limit is not empty [6].

A connected zero-dimensional set consists of one point [8].

A uniformly continuous function defined on a set that is everywhere dense in a compact, can be extended to a function over the entire compact. This extension is unique.

A reciprocal one-to-one, and similar (order preserving) correspondence between two sets s_1 and s_2, where s_1 is a denumerable everywhere dense subset of a segment I, and s_2 is a denumerable everywhere dense subset of a segment I_2, can be extended to a homeomorphism between the segments. Such an extension is unique.

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